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Can Leveraged Investments Increase Risk-Adjusted Returns over a Life Cycle?

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Abstract

This paper examines whether leveraging investments (buying on margin) when young can produce higher risk-adjusted returns over a life cycle. We use dual- and single-target leveraged investment strategies proposed in [Ayres and Nalebuff, 2008] and compare them to traditional all-stock and life-cycle investment strategies. We use a Monte Carlo approach to simulate market returns from different data-generating processes. Our focus is on returns drawn from a constant mean GJR-GARCH(1,1) model with skewed Student's t-distributed error terms to closely match the characteristics of empirical market returns.

We find that leveraged dual- and single-target strategies yield higher terminal wealth than unleveraged strategies in 95% of simulations. Additionally, the dual- and single-target strategies outperform traditional strategies on a risk-adjusted basis. This is evident from their higher average Sharpe ratios and lower relative risk premiums compared to an all-stock strategy. However, we find that 1.12% of investors default during our baseline investment horizon of 50 years, although this result is highly dependent on the data generating process chosen. Finally, the dual- and single-target strategies achieve higher temporal diversification by evening out the investor's exposure to the stock market over time. Combining these results, we recommend a dual or single-target strategy to young risk-averse Danish investors. Even if investors do not follow our preferred strategies, there is at least a benefit to investing early in life to obtain temporal diversification.

Contents

1 Executive Summary

Many investment strategies employ leverage to increase exposure to the stock market. Risk-neutral or even risk-seeking investors often use leverage to improve the expected returns of their portfolios. We explore the use of leverage as a tool for evening out exposure to the stock market and reducing risk over a life cycle.

To do this, we build a theoretical framework where risk-averse investors, characterized by a constant relative risk aversion (CRRA) utility function, maximizes their expected utility from wealth with respect to their allocation, π, into a risky asset. This problem has a well-known solution in the Merton fraction, where investors invest a constant fraction of savings into the risky asset.

$$\pi^* = \frac{1}{\gamma}\frac{\mu_y - \nu - r_i}{\sigma_y^2}, \tag{1}$$

which is independent of time and wealth and depends on risk aversion, γ and returns, μ_y and variance, σ_y^2 of the risky asset, as well as investment cost, ν.

Investors can invest in either a risky asset always paying the riskless rate, r_f, or the risky asset, S_t. We simulate a risky asset generated with a GJR-GARCH(1,1) process with skewed Student's t-distributed error terms for better empirical fit, as well as other data generating processes.

We test four different investment strategies. Two leveraged strategies: Dual-target and single-target strategies and two unleveraged, more traditional strategies: an all-stock strategy and a life-cycle strategy. The dual- and single-target strategy leverages investments when the investor is young until he reaches an initial set investment target. The investor then deleverages using savings and returns of his portfolio. Once deleveraged, the investor holds the optimal Merton-Samuelson portfolio. The all-stock strategy invests 100% of savings into stocks in all periods. The life-cycle strategy allocates 90% of savings into stocks when young, which linearly falls to 50% at age 65.

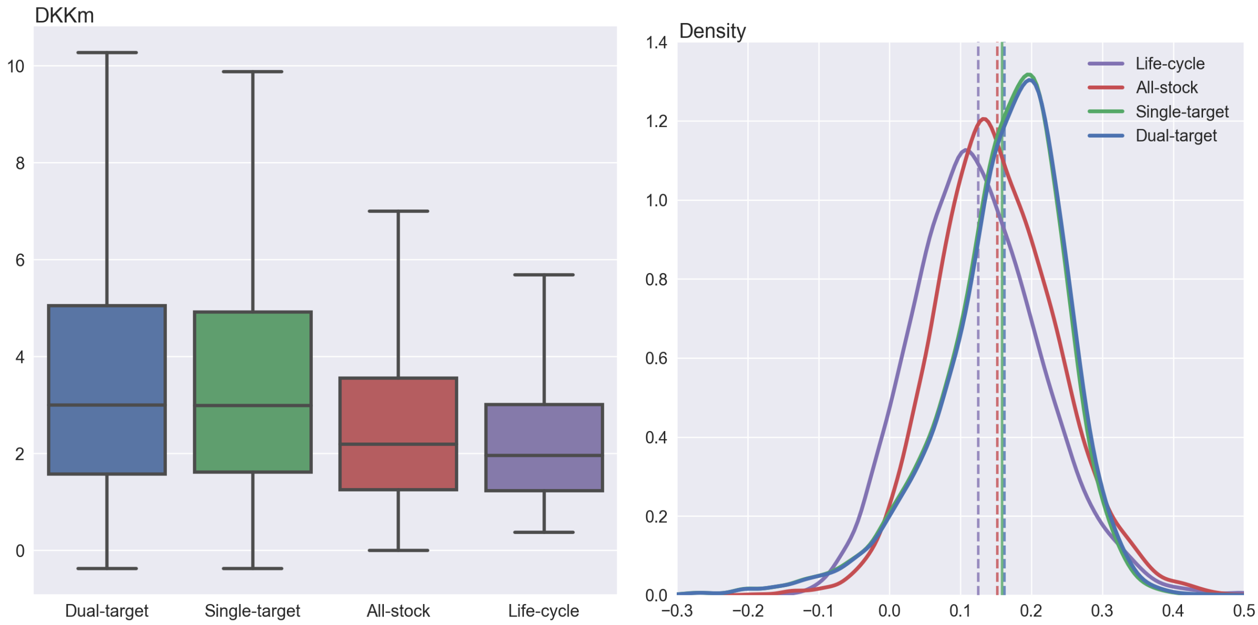
We evaluate the strategies using the average terminal wealth of the investor in each simulation. We calculate risk-adjusted returns using the Sharpe ratio and the relative risk premium over the certainty equivalent. The latter is the relative risk premium the investor demands over the amount that makes him indifferent between a certain dollar amount and the risky investment strategy.

We use the last 30 years of the S&P500 index as our risky asset, S_t, and fit the GARCH type model using it. The risk-free rate is the 15 year average of 10-year Danish government bond yields. We use a fixed 5% of the investors disposable income for savings, which follows an interpolated polynomial curve based on median disposable income data from Statistics Denmark for more realistic savings profiles.

We simulate markets for an investor with a 50-year investment horizon. The dual- and single-target strategies handily outperform the more traditional all-stock and life-cycle investment with 95% of simulations resulting in a higher terminal wealth. The boxplot of terminal wealth in figure 1a shows the differences in performance with higher 25^{th}, 50^{th} and 75^{th} quantiles of terminal wealth for the leveraged strategies. The dual- and single-target strategies differ minimally for the baseline model but note that this will change as we change the parameters of the baseline model.

Figure 1: Plots of terminal wealth

(a) Distribution of terminal wealth (DKKm) **(b)** Density of Sharpe Ratios



Note: Note that we have removed some observations with very high terminal wealth. Dotted lines on the Sharpe ratio plot indicate mean values.

The higher terminal wealth of the dual- and single-target strategies might be the cause of greater risk-taking. To check if this is indeed the case, we compute Sharpe ratios for all four strategies. In table 1, we see that the mean and median Sharpe ratio of the dual- and single-target strategies are higher than the all-stock strategy, all of which are higher than the life-cycle strategy. This implies that the dual- and single-target strategies produce higher returns per unit of risk taken. The distributions of Sharpe ratios of the dual- and single-target are skewed to the left, meaning that a few investors with abysmal performance pull the mean Sharpe ratio down. This is in contrast to the all-stock and life-cycle strategies, where we see the opposite.

The relative risk premiums show that investors demand the smallest relative risk premium of 50.13% over the certainty equivalent for the life-cycle strategy, thus deeming this the least risky compared to the all-stock strategy where investors demand a 132.86% risk premium. The dual- and single-target strategies lie between these. This shows that risk-averse investors prefer the outcomes of the leveraged strategies compared to the all-stock strategy.

In addition to simulations, we also test the strategies using Robert Shiller's historical market data. From 50-year rolling window backtests, we find that the leveraged strategies would have faired well on a risk-adjusted basis, judging by their higher Sharpe ratios compared to

Table 1: Average and [median] Sharpe Ratios and Relative risk premiums

	Dual target	Single target	All Stock	Life cycle
Sharpe Ratio	0.161 [0.174]	0.159 [0.171]	0.152 [0.146]	0.124 [0.119]
Relative Risk Premium	102.26%	87.76%	132.86%	50.13%

the unleveraged strategies. When comparing the strategies on average terminal wealth, the leveraged strategies only come out slightly on top. Importantly, zero investors would have defaulted during our sample period from 1871 to 2021.

All investors using a leveraged strategy expose themselves to default risk. We define defaulting in this setting as a case where the portfolio's total value (cash and stocks) is negative. By filtering out simulations where the investor defaults, we have analyzed what typically causes defaults. It usually happens when extremely negative returns hit the investor at the point of maximum leverage such that the investor cannot service his debt and eventually defaults. In our strategy, 112 investors out of 10.000 default resulting in a default rate of 1.12%. This default rate is heavily dependent on the data generating process of the returns of the stock. If we change the process to a more conventional fitted IID Normal distribution or 'draws with replacement,' the default rate drops to 0%.

The dual- and single-target strategies produce higher Sharpe ratios while using leverage. One might expect leverage to both increase volatility and average returns so that the Sharpe ratio should be roughly constant. However, we see that the Sharpe ratio is higher for the leveraged strategies. The higher Sharpe ratio is partially due to temporal diversification, where the investor spreads his exposure to stocks by shifting future savings into the present. Increasing exposure-smoothing reduces the risk of being overexposed late in life and underexposed early in life. The dual- and single-target strategies are 40% more exposed to stocks than the all-stock strategy in the first 15 years, a difference that slowly goes away as the investor deleverages.

The dual-target strategy is robust against increasing the margin rate, r_m, with only small drops in the 25^{th}, 50^{th} and 75^{th} quantiles of terminal wealth. The single-target strategy does far worse as its stock allocation depends on the margin rate for the entire horizon, whereas the dual-target switches to depend on the risk-free rate. Increasing the margin rate has a negligible effect on the default rate. When margin rates increase, the allocation in the risky asset falls such that the default probability is roughly the same. Both strategies are viable on both a shorter and longer investment horizon than the default 50 years with higher Sharpe ratios for the leveraged strategies over the unleveraged strategies for all tested horizons.

Changing the data-generating process of market returns has a substantial effect on the results. In our preferred GJR-GARCH(1,1) model, the Sharpe ratio of the leveraged strategies is higher than the unleveraged strategies. The outperformance only increases when using more conventional simulation methods like an IID fitted normal distribution or 'draws with replacement' of empirical S&P 500 returns. Likewise, changing the data-generating process lowers the relative risk premiums for all strategies dramatically. When we use an IID process to simulate returns, volatility clustering disappears, resulting in evenly distributed returns, which reduces the overall risk of the strategies.

Our recommendation for a young, danish student looking to invest his savings is to use a dual-target strategy. The dual-target strategy yields the highest mean and median terminal wealth and the highest Sharpe ratios in our simulations, giving the highest return per unit of risk. The leveraged strategies have a lower relative risk premium than an all-stock strategy. This recommendation rests upon the current margin rate, the risk-free rate, and expected average market returns, which will vary during the following decades and significantly affect the results. Additionally, an investor considering a dual-target strategy needs to consider the time and energy spend applying for loans and servicing debt. In any case, it is a good idea to start investing early, if just in a standard all-stock strategy. The benefits of time diversification are present regardless of using our leveraged strategy or not - the leveraged strategies merely intensify them.

2 Introduction

Optimal portfolio allocation is a field of research with potentially substantial effects on the quality of life of everyday citizens. Due to the compounding nature of investments, retirement wealth with an optimal investment strategy can be far greater than the retirement wealth of a sub-par investment strategy. What constitutes an optimal investment strategy is a topic of much debate and continuous research. The use of leverage, i.e., borrowing money to invest, has been researched for many decades with the allure of the idea described by [Samuelson, 1969]

> *"..., he [an investor] can look forward to a high salary in the future; and with so high a present discounted value of wealth, it is only prudent for him to put more into common stocks compared to his present tangible wealth, borrowing if necessary for the purpose."*

Leveraging investments might sound like a foreign idea. However, it is widely used in the housing market, serving as the primary pension savings for many people. Over a life cycle, the average investor could potentially unlock considerable diversification benefits by using leverage to invest in other asset classes, for example, stocks.

For life cycle investment, [Merton, 1969] developed theoretical continuous time models of varying time horizons for portfolio selections - famously computing the Merton fraction. This constant optimally determines what fraction of savings an investor should allocate to risky assets over a lifetime which, surprisingly, is independent of time and wealth.

[Viceira, 2001] states a version of Merton's investment problem in discrete and finite time. He finds that the solution to the discrete-time investment problem is identical to the original Merton Fraction. He shows that, depending on risk aversion, some investors prefer high levels of leveraged investment.

[Shiller, 2005] analyzes the US stock, bond, and money market from 1871 till 2004 to calculate returns and variances to compare different investment strategies. He finds that the more heavily invested in the equity market a given investment strategy is, the higher returns it earns. Furthermore, an all-stock strategy also generates higher wealth in the lower quantiles, indicating that the all-stock strategy also ensures higher returns than the other strategies, even when market returns are not favorable.

[Ayres and Nalebuff, 2008] extends Shiller's analysis by allowing for more than 100% stock exposure. The authors test investment strategies involving moderate usage of leverage when young and more conservative investments than an all-stock strategy when approaching retirement.

Moreover, they explore the benefits of temporal asset diversification, which means spreading an investor's exposure to risky assets over time. This reduces the risk of having too little (high) exposure when markets are favorable (unfavorable).

[Ayres and Nalebuff, 2008] device two leveraged investment strategies, a dual- and single-target strategy. Using stock market data from 1871 till 2005, they estimate for both strategies that investors with isoelastic preferences should invest between 169.4% and 5.6% of their savings in risky assets, depending on their level of risk aversion.

[Ayres and Nalebuff, 2008] also find that their strategies stochastically dominate both a lifecycle strategy and an all-stock strategy with 90% and 17% higher average retirement savings, respectively. Additionally, the authors explore how leverage costs and lower market returns would affect their results and find that their results are robust against changes from their base setup.

Their tests involve Monte Carlo simulations where returns are drawn with replacement from Robert Shiller's database of market returns since 1871. This is somewhat problematic as this removes clustering of returns, a stylized fact about financial time series — more on this in the data description section.

We apply the investment strategies suggested in [Ayres and Nalebuff, 2008] from the point of view of a young Danish student with a particular focus on the cost and availability of leverage, answering the question:

Does leveraging investments when young outperform other more traditional investment strategies on a risk-adjusted basis?

To do this, we investigate *is a leveraged investment strategy feasible for a young Danish student?* And if so, *does our preferred leveraged investment strategy diversify exposure to equities/stocks over time and thereby reduce risk?* And *under what market conditions for the next 50 years would our preferred leveraged investment strategy fail?*. To answer these questions fairly, we include taxation of the strategies' returns and interest rate tax deductions to reflect the real wealth effects young Danish investors face. We expand the Monte Carlo simulations of [Ayres and Nalebuff, 2008] by adding a more realistic data generating process driven by a constant mean GJR-GARCH(1,1) model with skewed Student's t-distributed error terms. We also test naive methods of drawing returns from a fitted normal distribution and a skewed Student's t-distribution. We compare the different methods and test whether they change the results.

To answer our research questions, we present theoretical arguments for optimal asset allocation for an investor with isoelastic preferences. He has a finite investment horizon, and two assets characterize the market. We test the different investment strategies in section 3. Section 4 argues for the superiority of a Student's t-distributed GJR-GARCH(1,1) model for market simulations over traditional methods like draws with replacement from an empirical sample or naively drawing from a Normal or Student's t-distribution. In section 6, we present empirical results from the simulations and perform various robustness checks to examine the strengths and weaknesses of different investment strategies.

We find that the dual and single-target investment strategies presented in [Ayres and Nalebuff, 2008], outperform the traditional investment strategies with higher terminal wealth for investors and higher Sharpe ratios using the GJR-GARCH(1,1) simulations. Additionally, we find a default rate of 1.12% for the 50-year investment horizon. By utilizing leverage, investors spread out their exposure to the risky asset, explaining the outperformance in the

Sharpe ratios. Considering these results, we recommend the dual- or single-target strategies to risk-averse, young Danish investors. Still, the practitioner should also consider the added time and effort it would take to follow this strategy compared to an all-stock strategy. More generally our results highlight the benefits of temporal diversification, underlining the importance of starting investing early.

3 Theoretical Section

In the following section, we develop the theoretical arguments for risk-averse investors seeking an optimal asset allocation between a risky and a risk-less asset. Additionally, we present a model for asset price dynamics and investment strategies. We also present risk-adjusted measures for portfolio performance.

3.1 Utility Theory

Investors have preferences regarding investment choices, primarily affected by expected returns and volatility. Many different utility functions model the trade-off between these two factors, where one of the more general utility functions is the Hyperbolic Absolute Risk Aversion (HARA) utility function given by

$$U(X_t) = \begin{cases} \frac{1-\gamma}{\gamma}\left(\frac{aX_t}{1-\gamma}\right)^{\gamma} & \gamma > 0 \\ \ln(aX_t + b) & \gamma = 0 \end{cases},$$

Here, investors have Linear Risk Tolerance (LRT), which these types of preferences are also called. The special cases of this utility function are more widely used, like the Constant Absolute Risk Aversion (CARA) utility function (exponential utility) where $b = 1$ and $\gamma \to -\infty$. This formulation is convenient mathematically but empirically less realistic than other special cases like the Constant Relative Risk Aversion (CRRA) utility function. With CRRA utility, investors have isoelastic preferences ([Ingersoll, 1987], p.37-41). This is the utility function we use to model risk-averse investors.

$$U(X_t) = \begin{cases} \frac{X_t^{1-\gamma}}{1-\gamma} & \gamma > 0 \\ \ln(X_t) & \gamma = 0 \end{cases},$$

where γ is the degree of constant relative risk aversion, which empirical studies put in a range from 1 to 4 with mean 2 ([Conine et al., 2017] table 1+2).
The investor decides on allocating his wealth, X_t, with a fraction π in a risky asset, S, using the following maximization problem as presented in [Viceira, 2001] but with a finite horizon allowing for a closed-form solution.

$$\max_{\pi_t} \mathbb{E}\left(\sum_{t=0}^{T} \delta^t U(X_t)\right),$$

subject to the following budget constraint

$$X_{t+1} = (X_t + Y_t - C_t)R_{t+1},$$

where C_t is consumption in period t. In order to focus our analysis, we assume that consumption C_t is chosen optimally ([Ayres and Nalebuff, 2008], p. 10). Y_t is wages earned in period t. R_{t+1} is the total return on all investments given by

$$R_{t+1} = \pi_t(1 + \mu_y - \nu) + (1 - \pi_t)(1 + r_i) \quad \text{for } i = f, m$$

where μ_y is the expected yearly return of the stock, S, and ν is the yearly investment fees in percent. r_i are different interest rates depending on the alternative the investor has. If the investor employs leverage, the alternative to the risky assets is the margin rate, i.e., the interest rate paid on the debt, r_m. If the investor is unleveraged, the alternative is the risk-free rate on safe assets r_f.

The optimal allocation, π^*, in the risky asset, S, is given by the Merton fraction in equation (2) as shown by [Viceira, 2001]. Here, an investor invests a constant fraction, π, of future savings into the risky asset, S, with this fraction being independent of time and wealth. Thus, the optimal allocation in the risky asset, π^*, is given by

$$\pi^* = \frac{a - r_i}{\gamma\sigma^2} = \frac{1}{\gamma}\frac{\mu_y - \nu - r_i}{\sigma_y^2}, \tag{2}$$

where σ_y is the yearly volatility of the stock and a is the yearly net returns ([Merton, 1969],p 253 , equation (29')).

3.2 Performance measurement

To compare the performance fairly, we have to measure it on a risk-adjusted basis. We do this in two different ways. First, by calculating the relative risk premiums, φ, of the individual strategies. Second, by calculating the mean monthly return and standard deviation per strategy and from this calculating the Sharpe ratio of each strategy.

The *certainty equivalent* is the terminal wealth W_{CE} that an investor is just as satisfied receiving with certainty as some risky terminal wealth W.

$$U(W_{CE}) = E[U(W)]$$

With N possible outcomes of the investment strategies all being equally likely with probability, $p = 1/N$

$$E[U(W)] = \sum_{i=1}^{T} p_i U(W_i) = \frac{1}{N}\sum_{i=1}^{T} U(W_i)$$

Inserting the CRRA utility function yields

$$\frac{W_{CE}^{1-\gamma}}{1-\gamma} = \frac{1}{N}\sum_{i=1}^{T}\frac{W_i^{1-\gamma}}{1-\gamma}$$

Resulting in the certainty equivalent W_{CE}

$$W_{CE} = \left(\frac{1}{N}\sum_{i=1}^{N}(W_i)^{1-\gamma}\right)^{1/(1-\gamma)}$$

The certainty equivalent can only be used to compare strategies with similar means ([Ingersoll, 1987],p.37-38). Our strategies might not have this, so we use the certainty equivalent to calculate *the relative risk premium*, φ, as

$$\varphi = \left(\frac{E(W)}{W_{CE}} - 1\right) \cdot 100$$

The relative risk premium, φ, describes the relative risk premium the risk-averse investor demands for taking the risky strategy.
Moving on to the *Sharpe ratio*, where we define the gross return with debt subtracted $r_{t,s,k}$ of strategy s for random seed k at time t as:

$$r_{t,s,k} \equiv \frac{W_{t,s,k}}{S_{t,s,k}}$$

Where $W_{s,k}$ is wealth (equity and cash) and $S_{s,k}$ is cumulative savings.
Then the average return $\hat{\mu}$ is:

$$\hat{\mu}_{s,k} \equiv \frac{1}{T}\sum_{i=2}^{T}\frac{r_{i,s,k}}{r_{i-1,s,k}} - 1$$

Similarly for the (sample) standard devation $\hat{\sigma}$:

$$\hat{\sigma}_{s,k} \equiv \sqrt{\frac{1}{T-1}\sum_{i=1}^{T}(r_{i,s,k} - \hat{\mu}_{s,k})^2}$$

Then using an approach by ([Hautsch and Voigt, 2019], p. 235), the average annual Sharpe ratio for strategy s is:

$$SR_s \equiv \frac{1}{K}\sum_{k=1}^{K}\frac{\hat{\mu}_{s,k}}{\hat{\sigma}_{s,k}\sqrt{12}}$$

The Sharpe ratio measures returns per unit of risk meaning, the higher the Sharpe ratio, the higher the risk-adjusted return.

3.3 Investment strategy

We test four investment strategies outlined by ([Ayres and Nalebuff, 2008],p. 10+17-18). For the leveraged strategies, the investor uses the Merton fraction in equation (2) to calculate his optimal allocation,$\pi(r_i)$ for $i = m, f$ into the risky asset. Note that r_m is the margin rate, and r_f is the risk-free rate. Switching between these changes the optimal allocation. The current empirical fraction allocated to the risky asset can either by measured against current wealth, CW, or some initial stock target, IST, being denoted $\hat{\pi}^{\text{CW}}(r_i)$ and $\hat{\pi}^{\text{IST}}(r_m)$ respectively. The investor compares his calculated target, $\pi(r_i)$, with these empirical fractions. The investor sets his initial stock target, IST, as his expected cumulative future savings times the fraction of wealth allocated to stocks, $\pi(r_m)$. Initially, the investor will be constrained as he only has his current liquid savings available, so leverage is used to get closer to the initial stock target,IST. Consider first the The *single-target* (three phase) strategy:

1. In phase 1, all liquid wealth is invested in the risky asset with target, $\pi(r_m)$ of IST, using a 1:1 level of leverage until $\hat{\pi}^{\text{IST}}(r_m) = \pi(r_m)$ is reached.

2. In phase 2, the investor deleverages the portfolio holding $\hat{\pi}^{\text{IST}}(r_m) = \pi(r_m)$ constant until the investor have no debt.

3. In phase 3, the investor holds the optimal portfolio using the Merton fraction,$\hat{\pi}^{\text{CW}}(r_m)$ of current wealth, until retirement. Rebalancing when needed.

We illustrate the strategy with a numerical example. Consider an investor with $\gamma = 2$, $\mu_y = 0.07$, $\nu = 0$, $r_m = 0.02$, $\sigma_y = 0.03$ and expected cumulative future savings of 1,000,000 DKK. Then, using the Merton fraction, his optimal allocation is $\pi(r_m) = 83.3\%$ in the risky asset and IST $= 833,333$. In phase 1 with savings of 500 DKK, the investor borrows 500 DKK and invests 1,000 DKK into the risky asset. This repeats every period until IST is reached. Then, in phase 2, the investor maintains a constant exposure of $\hat{\pi}^{\text{IST}}(r_m) = 83.3\%$ while using savings and returns to deleverage. When deleveraged, the investor progresses to phase 3 where he holds $\hat{\pi}^{\text{CW}}(r_m) = 83.3\%$ now of current wealth until retirement.
Consider now the *dual-target* (four phase) strategy, where phase 1 and 2 are identical to the single-target but differs with in phase 3 and 4:

1. In phase 3, the investor invest all liquid wealth into risky assets until $\hat{\pi}^{\text{CW}}(r_f) = \pi(r_f)$. Note that $\pi(r_m) < \pi(r_f)$

2. In phase 3, the investor holds the optimal portfolio using the Merton fraction,$\hat{\pi}^{\text{CW}}(r_f)$ of current wealth, until retirement. Rebalancing when needed.

To illustrate the dual-target strategy we continue where our previous example left of. Consider the case where $r_f = 0.015$ then in phase 3, the investor will change his target allocation to $\pi(r_m) = 91.7\%$ and once $\hat{\pi}^{\text{CW}}(r_f) = 91.7\%$ has been reached the investor progresses to phase 4 where he holds $\hat{\pi}^{\text{CW}}(r_i) = 91.7\%$ now of current wealth until retirement.
Similarly to [Ayres and Nalebuff, 2008], we also test two unleveraged strategies: An all-stock strategy, where the investor allocates 100% of liquid wealth into the risky asset until retirement. In the 90%/50% life-cycle strategy, the investor invests 90% of liquid wealth at age 21 in the risky asset. The fraction invested in the risky asset decreases linearly until it reaches 50% at age 65.

3.4 Diversification

Fundamentally diversification has two axes: asset and temporal diversification. The concept of asset diversification is familiar to any investor and is the principle of spreading (diversifying) investments into many different assets. This avoids the risk of a large loss due to a single asset performing poorly ([Hull, 2018],p.525-526).
The other type of diversification, temporal diversification, spreads risk, not over assets but over time by spreading the exposure to risky assets over a longer period. This might be even more important than asset diversification as asset returns in the same period are often highly correlated, whereas returns from year to year are often uncorrelated. As such, the potential diversification benefit. Consider an asset X with returns in period $t = 1, 2, ..., T$ with standard deviations $\sigma_1, \sigma_2, ..., \sigma_T$. A stylized fact about returns is that the variance of returns is correlated for short time horizons. But over longer time horizons, the variance of returns is asymptotically independent (weakly mixing) such that $\text{corr}(X_t, X_{t+h}) = 0$ for $h \to \infty$ ([Taylor, 2011],p.195-196). Thus, temporal diversification is reminiscent of asset diversification. Consider a portfolio of asset X being held by an investor in period t and $t+i$. This will have combined variance:

$$\sigma^2_{t,t+i} = \sigma^2_t + \sigma^2_{t+i} + \rho_{t,t+i}\sigma_t\sigma_{t+i}$$

By the stylized fact of asymptotic independence $\rho_{t,t+i} \to 0$ as $i \to \infty$ i.e. increasing the time between the returns, such that

$$\sigma^2_t + \sigma^2_{t+i} + \rho_{t,t+i}\sigma_t\sigma_{t+i} \to \sigma^2_t + \sigma^2_{t+i}$$

We see that, theoretically, the risk is reduced by temporally diversifying. However, temporal diversification can be difficult to achieve as investors obviously don't have all their future savings available at the beginning of their life but will instead gradually built up their portfolio, which (hopefully) will peak at some point before the investor liquidates his portfolio to consume when retired. This gradual built-up of investments often results in the investor having low exposure to equities early in life and heavy exposure later. His low temporal diversification causes a higher sensitivity to negative market returns at the later stages of his investment strategy. ([Ayres and Nalebuff, 2008],p.29-30).
This is why life-cycle investment is popular. The investor is less exposed to riskier equities as he gradually transitions his equity positions into bonds but at the cost of lower expected returns. The dual- and single-target strategies likewise counter uneven exposure and obtain more temporal diversification.

3.5 Market

The market in this paper is characterized by a risk-less asset, B, which pays the risk free rate, r_f, and a risky asset, S. The returns, r_t, of the risky asset, S, is the percentage change of the stock price, $r_t = \big[S_t/S_{t-1} - 1\big] \cdot 100$. To model its mean, we use a constant mean model. Each period t corresponds to one day so μ is the unconditional monthly mean return of r_t.

$$r_t = \mu + \epsilon_t. \tag{1}$$

Several approaches exist for simulating this financial time series. A commonly used yet naive approach is to draw ϵ_t from some *identical, independently distributed* (IID) distribution. This is often a Gaussian $\mathcal{N}(0,\, \sigma^2)$ distribution. The appealing feature of this model is that it corresponds to the empirical version of the standard theoretical model of a market governed by an underlying Brownian motion. However, it yields simulations with a terrible fit to the empirical distribution of ϵ. We give a detailed examination of possible densities in Section 4. Although drawing ϵ from an IID distribution is computationally straightforward, it yields inaccurate simulations, as empirical studies have shown that the errors terms, ϵ_t, of financial time series, are not independent of each other, i.e., $\text{cov}(\epsilon_s, \epsilon_k) \neq 0$ for $s \neq k$. Quite the contrary, the variance of ϵ_s and ϵ_k is often highly correlated in the short run, and therefore not IID.

We can address the problem of independence of returns by replacing the process that generates ϵ_t with the following GJR-GARCH(1,1) that [Glosten et al., 1993] extends the variance process with an extra effect from past negative residuals.

$$\epsilon_t = \sigma_t z_t \qquad z_t \sim \text{skewed} - t_{\eta,\lambda}(0, 1), \tag{2}$$

$$\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2 + \kappa\epsilon_{t-1}^2 I_{\{\epsilon_{t-1}<0\}} \qquad \omega > 0, \alpha, \beta, \kappa \geq 0, \tag{3}$$

where $t = 1, 2, ..., T$ and the initial values are taken as given. I is an indicator function which is one when the residual of the previous period, ϵ_{t-1}, is less than zero. Additionally z_t follows a skewed student's t distribution, which fits the empirical distribution better than a normal distribution[1]. In a GARCH type model, the conditional variance in period t is dependent on the variance and residuals from previous periods. α states how much the variance in period t is affected by a shock in the previous period. β is how persistent the variance is i.e. how long these shocks last and κ is an extra effect from negative residuals as markets are more sensitive to negative news. Many types of GARCH models exist like the EGARCH or AGARCH that one could also use.

3.6 Simulations

Using a skewed$-t_{\eta,\lambda}$ GJR-GARCH(1,1) model, we simulate market returns using Monte Carlo methods derived from a simple process.

1. Obtain Maximum Likelihood estimates of $\theta = (\mu, \omega, \alpha, \beta, \kappa, \eta, \lambda)$ using the observations in the information set for period $t-1$, $\mathcal{I}_{t-1}$.

[1] Additional information on the exact density can be found in the appendix

2. Estimate σ^2_{t+1} using equation (3), $\hat{\theta}$ and $\mathcal{I}_t$. Then draw $z_{t+1} \sim$ skewed $- t_{\eta,\lambda}(0,1)$ and compute $r_{t+1} = \mu + \epsilon_{t+1}$.

3. Iterate step 2 until r_{t+M} have been obtained.

4. Then calculate the cumulative returns during the period and multiply that with S_t obtaining S_{t+M}.

Simulating markets using a fitted Normal distribution, skewed Student's t-distribution or draws from replacement is a far simpler process:

1. Draw r_t from a Normal distribution $\mathcal{N}(\mu, \sigma^2)$ (or the other distribution presented) fitted to the original data

2. Iterate step 2 until r_{t+M} have been obtained.

3. Then calculate the cumulative returns during the period and multiply that with S_t obtaining S_{t+M}.

These processes are repeated, thereby drawing different realizations of z_t or r_t, resulting in a different realized market. We can thus simulate a market an arbitrary number of times - we chose 10.000 iterations. We apply the four investment strategies to all the simulated markets using techniques from dynamic programming in Python, see appendix for the source code.

4 Description of data

This section presents the savings profile of our representative investor along with realizations of different market simulating methods.

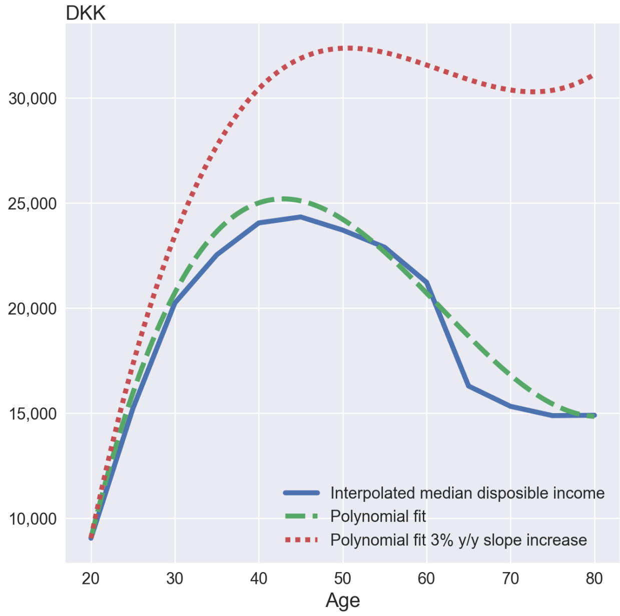
4.1 Savings profile

To model a realistic lifetime disposable income profile of an average Danish investor, we use the median disposable income for different age cohorts from Statistics Denmark. For ease of computation, we fit a polynomial curve to the data, see figure 2, to quickly generate disposable income profiles with differing time horizons, which will become helpful in the analysis [Danmark Statistisk, 021a][2].

If we used this disposable income profile in our simulations, we would assume zero income growth over and above what can be explained from the investor's age. This is problematic when we want a realistic simulation. To compensate for this, we fit a new curve with an increase to the slope of 3% y/y. This increase matches the yearly rise in nominal disposable income of about 3% for the last 25 years (INDKP201). This increased curve is the one we use as the baseline disposable income profile for our simulations.

[2] See Appendix A.3 for the fitted polynomial

Figure 2: Baseline lifetime monthly disposable income used in simulations



4.2 Market data

The data we use to fit our four simulation methods is a sample of daily returns of the S&P500 index from 4$^{\text{th}}$ of April 1991 to 1$^{\text{th}}$ of January 2021.

When simulating returns, the density of simulated returns must match the empirical density of S&P500 returns as these are the best estimate we have of the future. Suppose we fail to realistically model markets with significant crises. In that case, we could potentially bias our result in favor of the leveraged strategies, as we expect market crises to have the most prominent adverse effects on the leveraged strategies, potentially causing some investors to default.

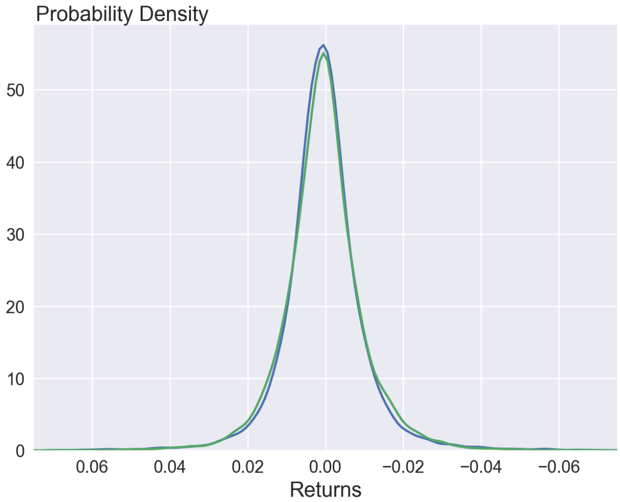
We now investigate how closely the different distributions fit the empirical data. In figure 3, we have plotted four simulated densities against the empirical density of S&P500 returns. In panel (c), we present draws from the normal distribution, and it is clear that the empirical returns are non-normal. Thus, any simulated returns from a normal distribution will provide inaccurate results in this setting.

In panel (a) and (d), draws from a skewed Student's t-distribution are visualized, with the only difference being the volatility structure of panel (a) being a GJR-GARCH(1,1). This, however, should not change the distribution of the simulated returns, which fit the empirical density much better than the normal distribution in panel (c) as panel (a) and (d) exhibits fatter tails and excess kurtosis.

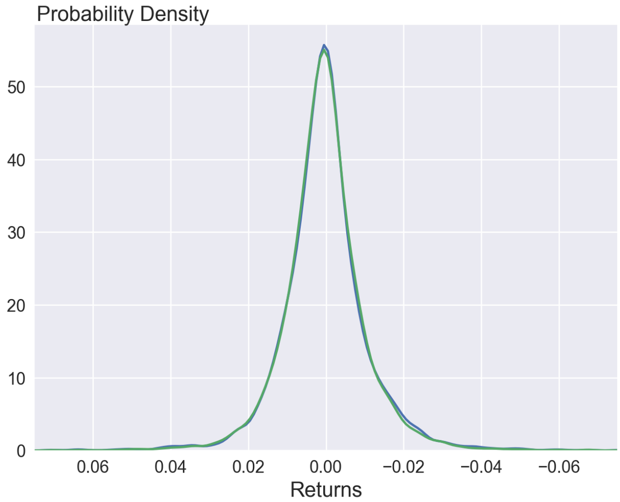
Finally, panel (b) is 'draws with replacement' from the empirical distribution, which unsurprisingly fits the empirical distribution perfectly. From a pure density point of view, draws with replacement seem like the best option, very closely followed by the skewed Student's t-distribution in panel (a) and (d). However, 'draws with replacement' has certain drawbacks,

Figure 3: Density of simulated returns and empirical density

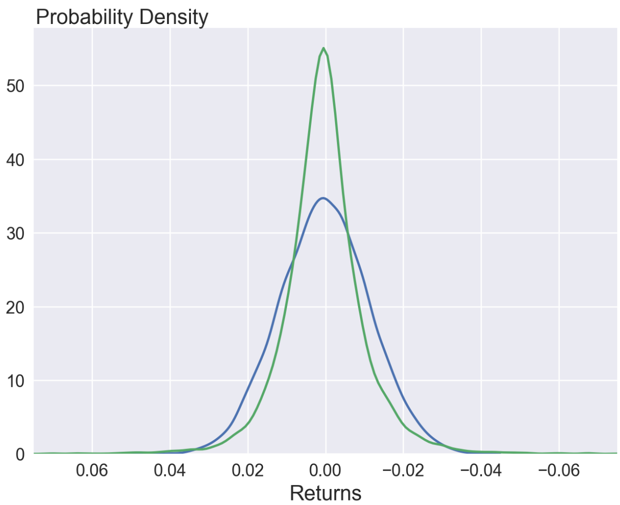
(a) skewed $- t_{\eta,\lambda}$ GJR-GARCH(1,1)



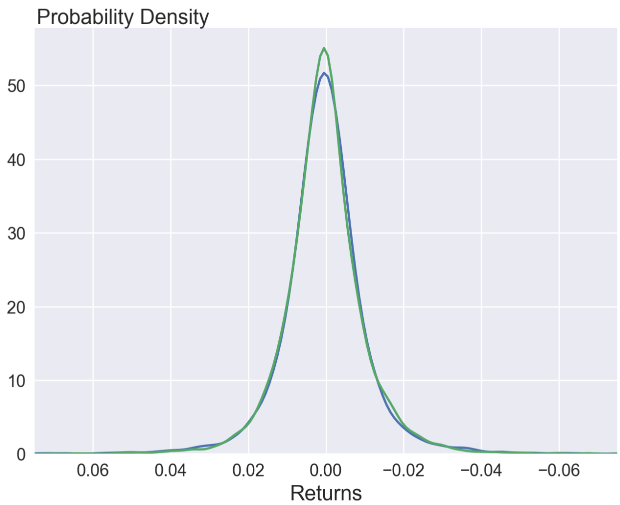
(b) IID draws with replacement



(c) IID Normal distribution



(d) IID skewed $- t_{\eta,\lambda}$-distribution



Note: The blue line in all panels are a realization of some distribution and will vary very slightly from realization to realization

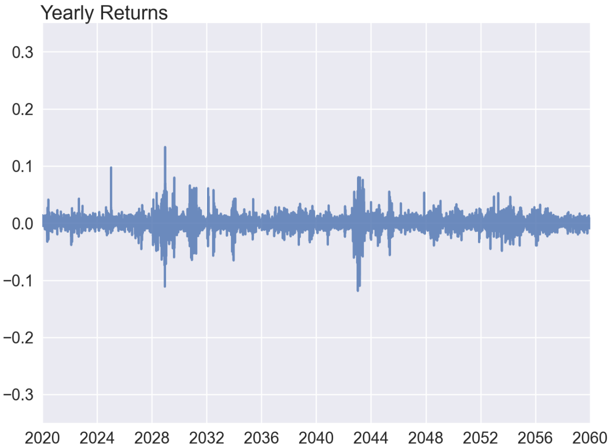
which we visualize in figure 4.

In figure 4, we see a random realization of data drawn from different distributions similar to those in figure 3 and the actual returns of the S&P500 index for a similar length period in panel (e).

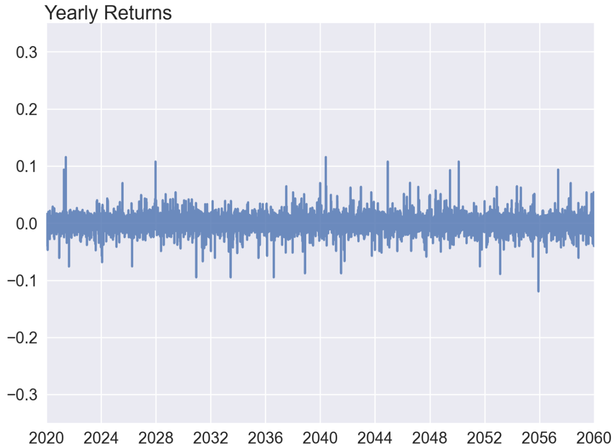
From panel (c), it is still apparent that IID draws from a normal distribution does not match the returns of the S&P500 index both in terms of size and the correlation. In panel (e), we see clustering of returns, i.e., small returns tend to be followed by small returns and large returns tend to be followed by large returns of either sign. We do not observe this in panel (c), meaning that draws with replacement lose this important empirical feature of returns.

Figure 4: Realization of simulations

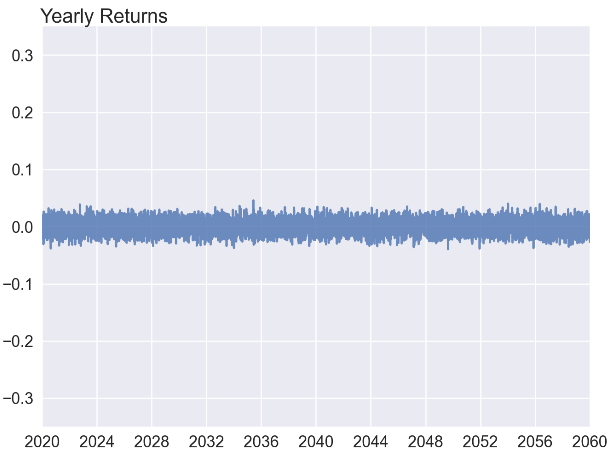
(a) skewed $- t_{\eta,\lambda}$ GJR-GARCH(1,1)



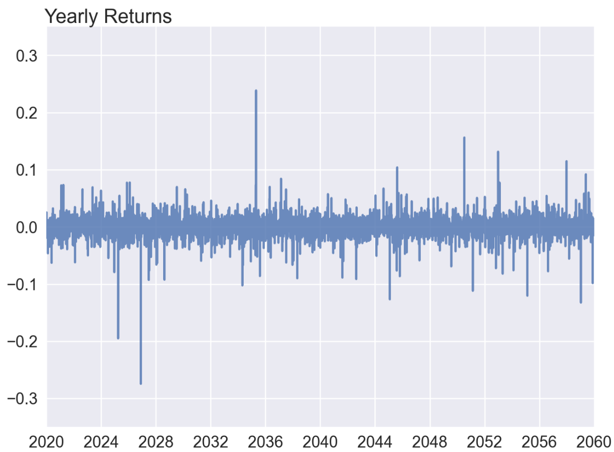
(b) Draws with replacement



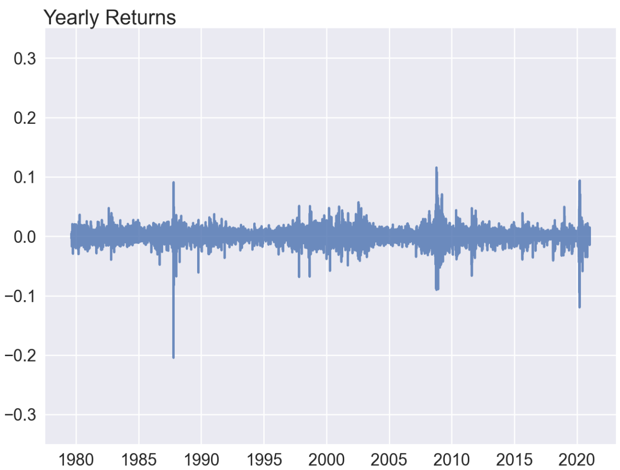
(c) IID Normal distribution



(d) IID skewed $- t_{\eta,\lambda}$-distribution



(e) Empirical S&P500 returns



Note: For panel (a)-(d), the data are a realization of some distribution and will vary from realization to realization. We have selected a realization that we thought best depicted an average realization

The same pattern holds for (b) and (d), where the size of the returns fits the actual series in

panel (e) but the correlation of returns does not as in panels (b) and (d). Here, returns are drawn IID and thus have no correlation, which is clearly present in panel (e) where we see clustered returns, most visible around the financial crisis in 2008-2009 and the dot com bubble in 2000-2001.
However, in panel (a), the magnitude of returns match panel (e), and we observe clustered returns such that the correlation of returns is conserved. This is why we use simulated markets drawn from a GJR-GARCH(1,1) model.
For both figure 3 and 4, one should note that all the data is just one realization of the process (except figure 4 panel e), and the data will vary from realization to realization. The important part is the statistical properties of the original time series and less whether it looks exactly like the original data.

5 Parameterization

This section explores the types of leverage available to the average young investor and the realistic taxation of capital gains. From this study of real-world restrictions, we calibrate a baseline model.

5.1 Possible avenues for leverage

One of the difficulties of a leveraged investment strategy, especially when young, is access to cheap and plentiful leverage. Financial institutions are often unwilling to lend a sufficient amount of money at interest rates where a leveraged strategy is viable. However, there exist several ways a young investor in Denmark can access cheap and somewhat plentiful leverage. For investors taking any education, one of these ways is to apply for 'SU-lån,' which is government-issued loans to students, mainly to cover living expenses. Still, one can equally use the money for investment if one's situation allows it. These loans are paid out monthly. As of writing, the maximum monthly rate is 3.234 DKK, which, if used every month, yields 38.808 DKK a year [Uddannelses- og Forskningsstyrelsen, 021a]. During education, the loan's interest rate is 4% per year and 1% after ending the education [Uddannelses- og Forskningsstyrelsen, 021b].
A second option is to use portfolio loans, which are loans issued by investment brokers directly for leveraging investments with or without security in the assets in the portfolio. Different banks have different terms, but the most attractive we could find was the portfolio loans offered by the popular broker Nordnet. They offer loans with a lower interest rate, the lower the amount borrowed, and the more diversified the portfolio is. Consider an investor with a well-diversified portfolio. Then the investor can borrow up to 60%(40%) of the invested amount at a 3% (2%) interest rate [Nordnet AB, 2021].
These two options seem the easiest and, more importantly, cheapest for our target practitioner and will be the ones we apply in our simulations. Combining the 3.234 DKK per month at a

4% interest rate and portfolio loans at a 2% or 3% interest rate should be enough to achieve a sufficient level of leverage to obtain a 1:1 ratio. Thus, we conclude that a leveraged investment strategy is indeed practically feasible.

Naturally, there are many different possibilities for leverage, and investors should investigate what options are available to get the best conditions possible. Below, we have listed additional options.

If the investor owns real estate, he can take out a mortgage, also called an Equity Home Loan. This option is probably not relevant early in life for our target applicant, but it may later. Mortgage debt is relatively cheap because the loan is collateralized. Many Danish mortgage banks currently offer a mortgage with a fixed interest rate from 1% to 1,5% for a 30-year mortgage at a favorable bond price. However, including a housing market is beyond the scope of this paper, so we neglect this option.

Different from direct leveraging is to invest in leveraged derivatives, such as leveraged exchange-traded funds (ETF's). These products have the advantage that an investor doesn't need to borrow money outright but instead invest in an asset where some leverage ratio, typically 1:1, is automatically applied. Though these are often cheaper than leveraging outright in terms of the yearly costs compared to the margin rate of loans, the drawback is that leveraged ETFs need to rebalance daily to match the targeted index using derivatives like swaps. In the long run, this will cause tracking errors compared to the underlying index, so investors should proceed with caution when investing in leveraged ETF for long-held positions. See [Lu et al., 2009] for a detailed discussion of this. For these reasons, we avoid leveraged ETF's when building our strategy.

Table 2: Overview of types of leverage

Type	**Amount**	**Interest rate**	**Term and Conditions**
SU Loan	3.234 DKK Monthly 38.808 DKK annually	4% during study 1% after study	Must be a student Must be repaid after ended education
Portfolio Loan	40% of invested amount 60% of invested amount	2% 3%	Max ind. asset weight 20% Max ind. asset weight 25%
Home Equity Loan	Max 80% of real estate	0.5%-1.5% and other fees	10-30 year duration
Bank overdraft	∼50.000 DKK	>5%	Varies among banks
Leveraged ETF	Unlimited	0.5-1% + fees	For short term investment Not for retail investors

Notes: See the written text in section 5.1 for sources

5.2 Taxation

We assume that the investor pays taxes on unrealized gains[3] because this is how ETFs are taxed in Denmark. Due to the high cost of domestic funds, we only consider foreign ETF's that track the S&P 500 index as a possible investment. These ETFs have yearly fees of around 7bps compared to around 50bps for the cheapest domestic funds. Since all ETF's are taxed on both realized and unrealized gains in Denmark, this is what we implement in our calculations. In Denmark, capital gains from stocks and bonds are taxed differently. Stocks are taxed at a special equity tax rate of 27% paid on the first 56.600 DKK and 42% for capital gains higher than this. Bonds or interest gains are taxed as capital income and are taxed as regular income with a marginal tax rate of 42.7%.

Additionally, capital losses are tax-deductible such that losses one year can be saved and used as a deductible when the investor has capital gains in the future. Especially relevant for the leveraged strategies is interest rate deductions, where 20.6% of an investor's interest payment is deductible. We note that interest rate payments from SU loans during studies are non-deductible. In contrast, they are deductible after the investor has graduated, i.e., when the investor repays the loan [Skattestyrelsen, 2021].

5.3 Baseline Model

The baseline model is our best estimate of the most realistic parameterization of the model to yield the most realistic outcome of the different investment strategies. We implement a gearing cap of 1:1, i.e., an investor can borrow a maximum of 1 DKK for every 1 DKK of savings he provides. We estimate the parameters of the Merton rule with the following empirical data. The expected yearly returns based on historical returns μ_y is 7.6%. We set the risk-free rate r_f as the average effective yield of danish government bonds for the past 15 years, which is 2% [Danmark Statistisk, 021b]. The margin rate r_m is the average percentage paid on our two kinds of leverage, which is 2.3%. The yearly investment fee, ν is given as 0.2%, and lastly, the volatility of the stock,σ_y^2 is given as the empirical variance of the S&P500, which is 0.0284. Finally, we set the level of constant relative risk aversion γ to 2. This allows us to calculate $\pi(r_m)$ and $\pi(r_f)$ as

$$\pi(r_m) = \frac{1}{2}\frac{0.076 - 0.002 - 0.023}{0.0284} = 93.41\% \qquad \pi(r_m) = \frac{1}{2}\frac{0.076 - 0.002 - 0.02}{0.0284} = 98.7\%$$

The skewed $-t_{\eta,\kappa}$ GJR-GARCH(1,1) parameter estimates are found via Maximum Likelihood estimation, which is given as

$$r_t = \underset{(3.606)}{0.0311} + \epsilon_t$$

$$\sigma_t^2 = \underset{(5.879)}{0.0154} + \underset{(0.0001)}{0.001}\epsilon_{t-1}^2 + \underset{(83.066)}{0.8966}\sigma_{t-1}^2 + \underset{(9.779)}{0.1831}\epsilon_{t-1}^2 I_{\{\epsilon_{t-1}<0\}}$$

The parameter estimates for the skewed Student's t-distribution, are $\eta = 7.085(12.052)$ and $\lambda = -0.1048(-7.104)$ with t-values reported in parenthesis.

[3]In Danish 'lagerbeskatning'

To be in line with [Ayres and Nalebuff, 2008], we assume that investors invest 5% of their disposable income, where the growth of income follows an interpolation of disposable income by age, such that the amount to be invested each period follows a realistic development [Danmark Statistisk, 021a].

6 Empirical Section

This section presents empirical results of the different investment strategies for the baseline model. We also explore various changes to the baseline model parameters to test the robustness of the strategies. In addition, we investigate the risk and benefits associated with the dual- and single-target strategies.

6.1 Results from the Baseline model

In table 3, we report the quantiles of terminal wealth and levels of π from the Monte Carlo simulations of the GJR-GARCH(1,1) simulations. We note that the median terminal wealth of the dual- and single-target strategies are nearly identical. Both are 37% higher than the all-stock strategy and 53% higher than the life-cycle strategy. In fact, in 95% of simulations, the dual- and single-target strategies yield higher terminal wealth than the all-stock strategy. We

Table 3: Terminal wealth of strategies from Monte Carlo simulations

	Dual-target	Single-target	All-Stock	Life-cycle
Initial π	93.41%	93.41%	100%	90%
Final π	98.70%	93.41%	100%	50%
Mean	3,838,926	3,733,564	2,708,186	2,378,441
10th pct.	792,156	827,907	701,323	834,364
25th pct.	1,578,737	1,619,017	1,253,909	1,236,068
Median	3,005,684	2,996,312	2,191,834	1,965,585
75th pct.	5,058,007	4,923,404	3,555,740	3,018,214
90th pct.	7,912,987	7,578,566	5,274,542	4,401,267
Defaulted %	1.12%	1.12%	0%	0%

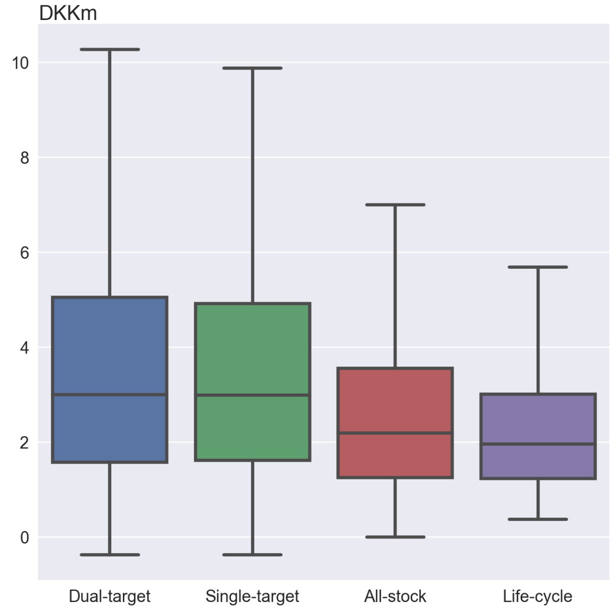
Notes: The results are based on 10.000 Monte Carlo simulations.

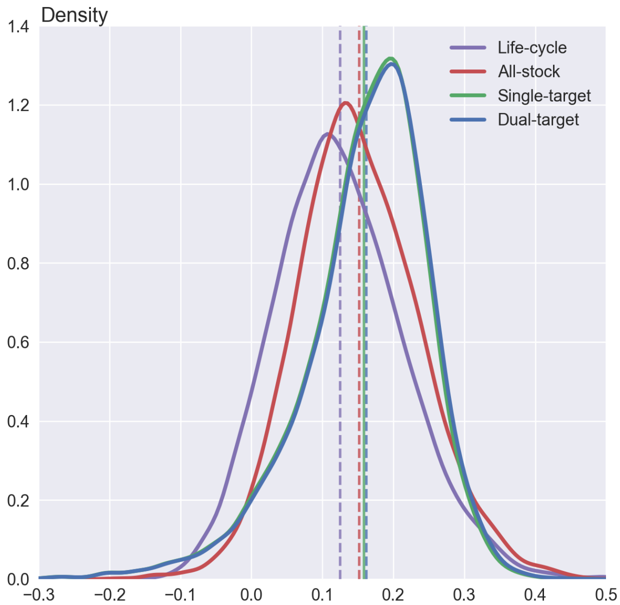
see a similar picture in the box-plot figure 5a. Here, the dual- and single-target strategies have higher 25^{th}, 50^{th}, 75^{th} and 90^{th} quantiles than the all-stock and life-cycle, and are roughly equal to the life-cycle strategy in the 10^{th} quantile. However, the dual- and single-target strategies have a default-risk as 112 out of 10.000 investors default, which we will explore in section 6.4.1 These results contrast to [Ayres and Nalebuff, 2008] who found that the dual and single-target strategies stochastically dominate the all-stock and life-cycle strategies. As we will discuss in section 6.2.4, this is partially due to differences in simulation methods.

Figure 5: Strategy performance in the baseline model

(a) Distribution of terminal wealth

(b) Density of Sharpe Ratios





Note: Note that we have removed some observations with very high terminal wealth in the box-plot. Dotted lines on the Sharpe ratio plot indicate mean values.

The dual- and single-target strategy for the baseline model have very similar outcomes with the 25$^{\text{th}}$, 50$^{\text{th}}$ and 75$^{\text{th}}$ quantile being very similar. These two strategies differ in figure 5a and table 3 where there is a general tendency for the single-target strategy to be more centered around the median than the dual-target strategy. These differences result from the dual-target strategy changing its allocation to the risky asset in phase four to 98.7% compared to 93.41% in the single-target strategy. Thus, the volatility of the dual-target strategy is higher than in the single-target strategy. I.e., in simulations where the single-target strategy performs poorly, the dual-target performs even worse and vise versa for good performances. This result is also evident from the box plot in figure 5a where the dual-target (blue) spans a greater distance from minimum to maximum than the single-target (green).

Table 4: Average and [median] Sharpe Ratios and Relative Risk Premium (%)

	Dual-target	Single-target	All-stock	Life-cycle
Sharpe Ratio	0.161[0.174] (0.90)	0.159[0.171] (0.90)	0.152[0.146] (0.88)	0.124[0.119] (0.92)
Relative Risk Premium	102.26%	87.76%	132.86%	50.13%

Note: Standard errors of the Sharpe Ratios in (·)

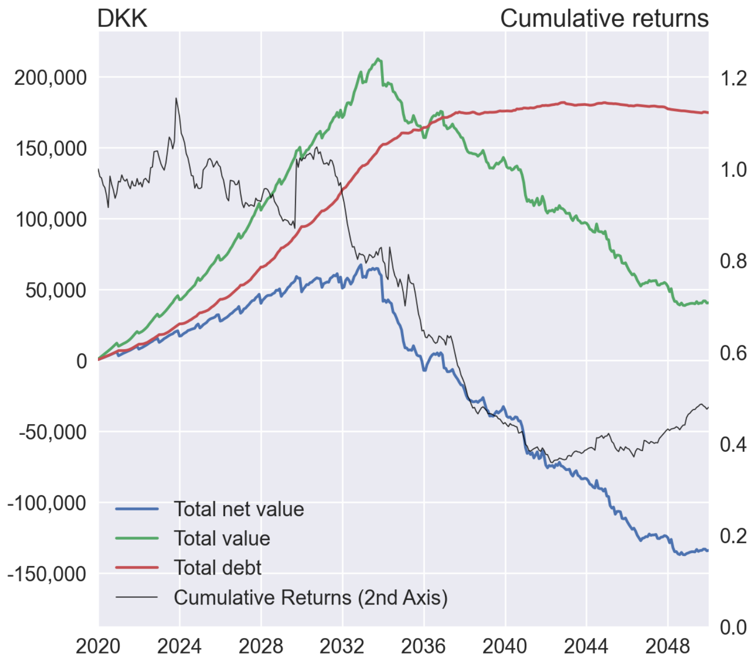
In table 4, we report two measures of risk-adjusted performance. Again, we note almost identical average Sharpe ratios of the dual- and single-target strategies at 0.161 and 0.159, re-

spectively. Both outperform the all-stock strategy and the life-cycle strategy, yielding average Sharpe ratios of 0.152 and 0.124, respectively. However, the median Sharpe ratio is noticeable higher than the average for the dual- and single-target strategies. More investors end up with negative terminal wealth, thus skewing the distribution to the left, as shown in figure 5. In conclusion, the leveraged strategies yield a better risk/reward payoff than the unleveraged strategies, both on average and on the median.
In table 4, we see that investors demand the highest relative risk premium of 132.86 % for the all-stock strategy. At the other end, investors unsurprisingly demand the lowest relative risk premium of 50.13% for the conservative life-cycle strategy, with the dual- and single-target strategies in between these two.
To summarize, the dual- and single-target strategies produce higher terminal wealth than the all-stock and life-cycle strategies, in the vast majority of simulations, with only a negligible risk of default. The dual- and single-target strategies are pretty similar, with the dual-target out-performing the single-target at the cost of slightly greater risk. The leveraged strategies yield a higher average and median risk-adjusted return than the life-cycle strategy, but investors demand a greater relative risk premium.
We will explore the default risk and the reasons for the leveraged strategies' out-performance in the following two subsections.

6.1.1 Default risk

Any investor employing a leveraged strategy is liable to default. In an unleveraged strategy, the investor can only lose the invested amount. In contrast, an investor using a leveraged investment strategy can lose the borrowed amount and the amount the investor invested out of pocket. We assume that the investor has defaulted when his cash and equity positions become worthless.
Using this definition, 112 out of the 10.000 investors default in our simulations during the entire 50-year investment horizon. The default rate of 1.12% is the same for the dual- and-single-target strategy, as investors have paid off their debt when the strategies diverge. This result is in contrast to [Ayres and Nalebuff, 2008] who found that no investor defaulted in their simulations. A default rate of 1.12% is in itself remarkably low considering the quite aggressive 1:1 leverage cap. But, compared to the all-stock and life-cycle strategies default rate of 0%, defaults are a definite drawback for the dual- and single-target strategies.
To explore what the typical market looks like for a defaulted investor, we take the average outcome of the 112 cases and plot the results in figure 6. We plot the mean of debt, total value of cash and equity positions, and total value minus total debt (Total net value). Additionally, the mean cumulative gross returns are plotted on the secondary axis. We see that the average defaulted investors are fully leveraged around 2035 and have positive portfolio values. Then the investors experience highly negative market returns, which is evident by the cumulative return falling drastically around 2032-2040, causing the net value of the portfolio to plummet.

Figure 6: Average strategy development for defaulted investors following the dual-target strategy



Note: Only the 112 investors who defaulted are plotted in this graph and the last 20 years are cut away

This, in turn, raises their interest costs as much less collateral covers the outstanding debt, meaning the investors are effectively more aggressively leveraged. The investors cannot pay off the debt (red line), neither by selling the risky asset nor by using their savings. Thus, they will default immediately following the negative market shock or slowly over time as the portfolio is sold off to repay the debt.

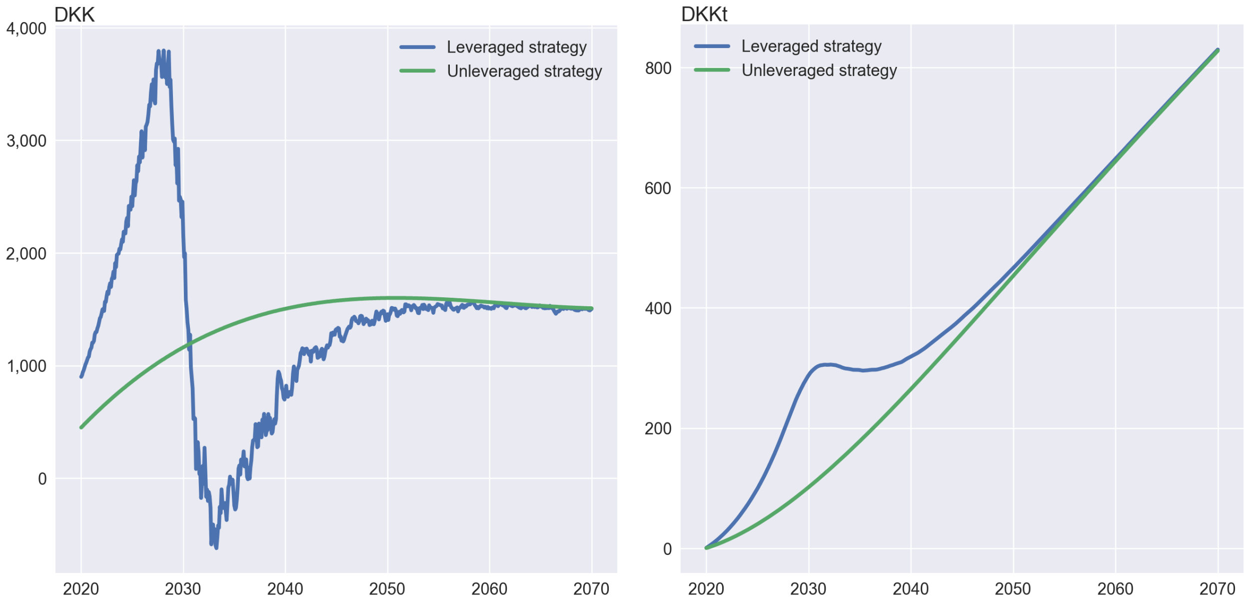
To summarize, extremely adverse market developments and very unlucky timings are needed for an investor to default. If, on the contrary, the investor is hit with a very negative return when his debt is repaid, the investor only loses parts of his equity position, similar to the all-stock and life-cycle strategies, and will not default.

6.1.2 Temporal diversification

Does the dual-target strategy achieve temporal diversification? One way to gauge this is to look at how much money the investor allocates to the risky asset on average across time. We have plotted both the monthly net investments and the cumulative version of the same data in figure 7. As the dual- and single-target strategies are very similar in net investments, we only plot the dual-target strategy. Net investments for the unleveraged strategies are the baseline savings since no leveraged is used. In the plot, the difference between the leveraged and unleveraged strategies is clear. Money is borrowed during the first ten years to increase stock exposure at a young age in the leveraged strategy. Quickly after that, the loans are being repaid. We note that the unleveraged strategy takes about ten more years before it catches up with the total level of equity in the levered strategy. So, in essence, leverage allows the investor to get a 10-year head start on stock exposure relative to the unleveraged strategy. The leveraged investor has a more even exposure towards stocks and is thus more temporally diversified than the unleveraged investor.

Figure 7: Empirical temporal diversification

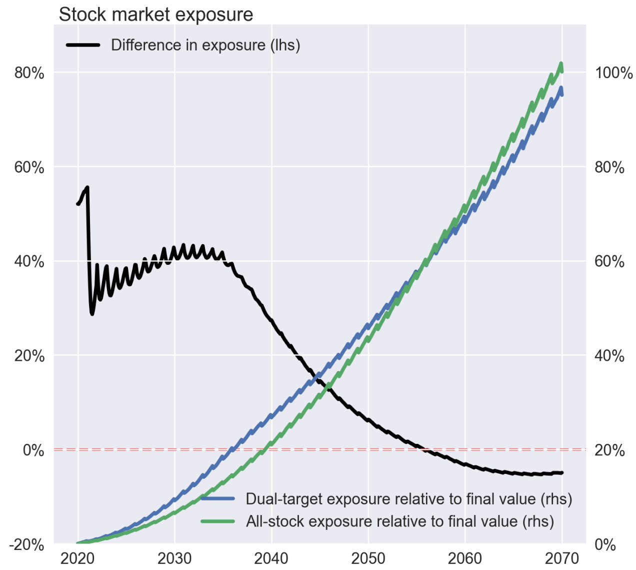
(a) Net monthly investments **(b)** Cumulative sum of net monthly investments



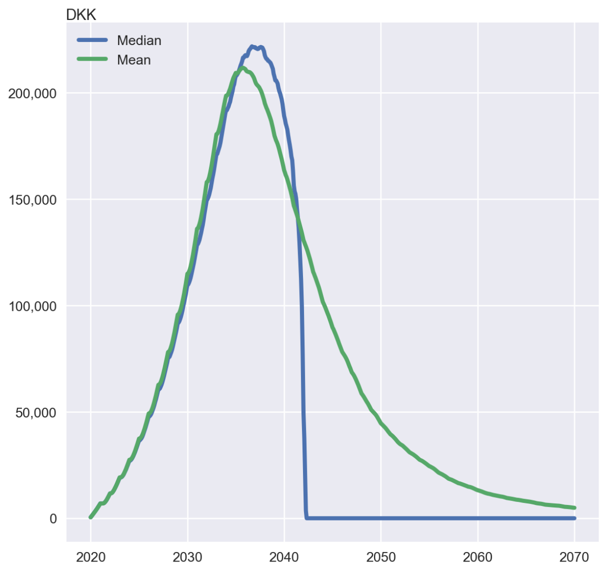
Another way to assess temporal diversification is by plotting the average stock market exposure across time. We define stock market exposure in the individual simulations as portfolio value divided by terminal wealth. This means all simulations of the all-stock strategy, by definition, end in 100% stock market exposure. In contrast, the dual-target strategy ends slightly below 100% as the target stock market exposure $\pi(r_f) = 98.7\%$. We have plotted this in figure 8 along with the difference in stock market exposure in percentage. To reduce clutter, we omitted the single-target- and life-cycle strategy. In the average simulation, we see that the dual-target strategy does spread stock market exposure across time compared to an all-stock strategy. After the first tax payment, the difference is about 40% more stock market exposure for the first 15 years. Afterward, the difference fades out, and the investor eventually ends up less exposed than in the all-stock strategy. This is only possible because of the debt obtained by the investor in the leveraged strategies. We see in the graph to the right that total debt peaks around 2035. This is around the same time the difference in market exposure is the highest, between the dual-target strategy and the all-stock strategy. Does this mean the leveraged investor obtains reduced risk compared to an all-stock strategy? Seemingly, yes. The benefits of temporal diversification seem to outweigh the mechanically increased risk that comes with higher leverage. As seen in figure 5 the average Sharpe Ratio is clearly higher for the leveraged strategies, meaning the risk/reward trade-off is better than the all-stock strategy. One possible explanation for this could be the increased temporal diversification.

Figure 8: Underlying mechanism of temporal diversification

(a) Difference in stock market exposure



(b) Debt progress in the dual-target strategy



6.2 Variations from the Baseline Model

The results presented in the section above come from our baseline model parameters. But, to be thorough, we will now consider variations of the parameters to test the robustness of our results and examine its strengths and weaknesses. Specifically, we vary the risk aversion of investors, the margin rate, the investment horizon, and finally, the data generating process used for simulating market returns.

6.2.1 Level of Risk aversion

A vital component of the Merton fraction is the investor's level of relative risk aversion, γ. Recall that the higher the level of relative risk aversion is, the lower the optimal fraction invested in the risky asset, π^*, is and vice versa by equation (2). Increasing the relative risk aversion means that the investor will allocate less of his wealth to the risky assets, as he is now more risk-averse. In figure 9, we see this, as the fraction invested in stocks $\pi(r_i)$ for $i = m, f$, decreases exponentially as γ increases.

Changing the relative risk aversion will naturally affect the dual- and single-target strategies but not the all-stock and life-cycle strategies. We expect investors with higher risk aversion to earn lower terminal wealth on average and lose less in bear markets. The lower allocation into the risky asset naturally changes the terminal wealth of investors quite drastically, as seen in figure 10. The 25$^{\text{th}}$, 50$^{\text{th}}$ and 75$^{\text{th}}$ quantile all fall as risk aversion increases due to lower allocation in the risky asset. This also reduces default risk with 112, 76, 72, and 59 investors defaulting for γ being 2, 3, 4, and 5, respectively, which aligns with the investors' higher risk aversion. Lastly, consider the performance measurements in table 5, where the Sharpe ratio decreases as relative risk aversion, γ, increases due to lower allocation into the risky asset.

Figure 9: How does π change with the level of risk aversion, γ

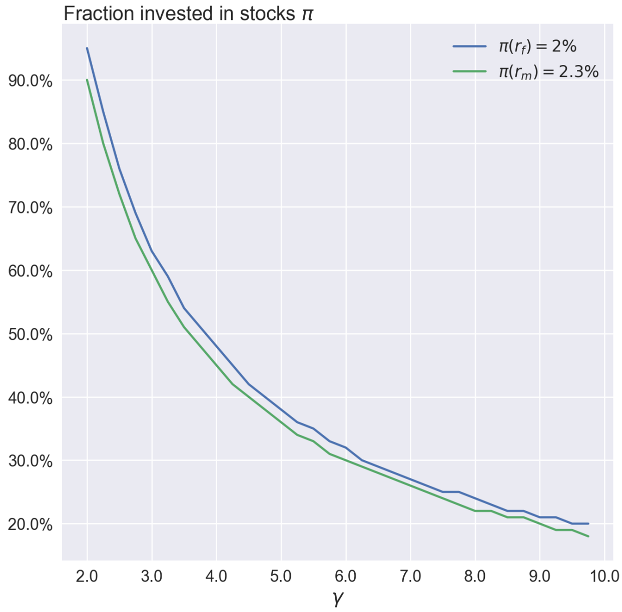
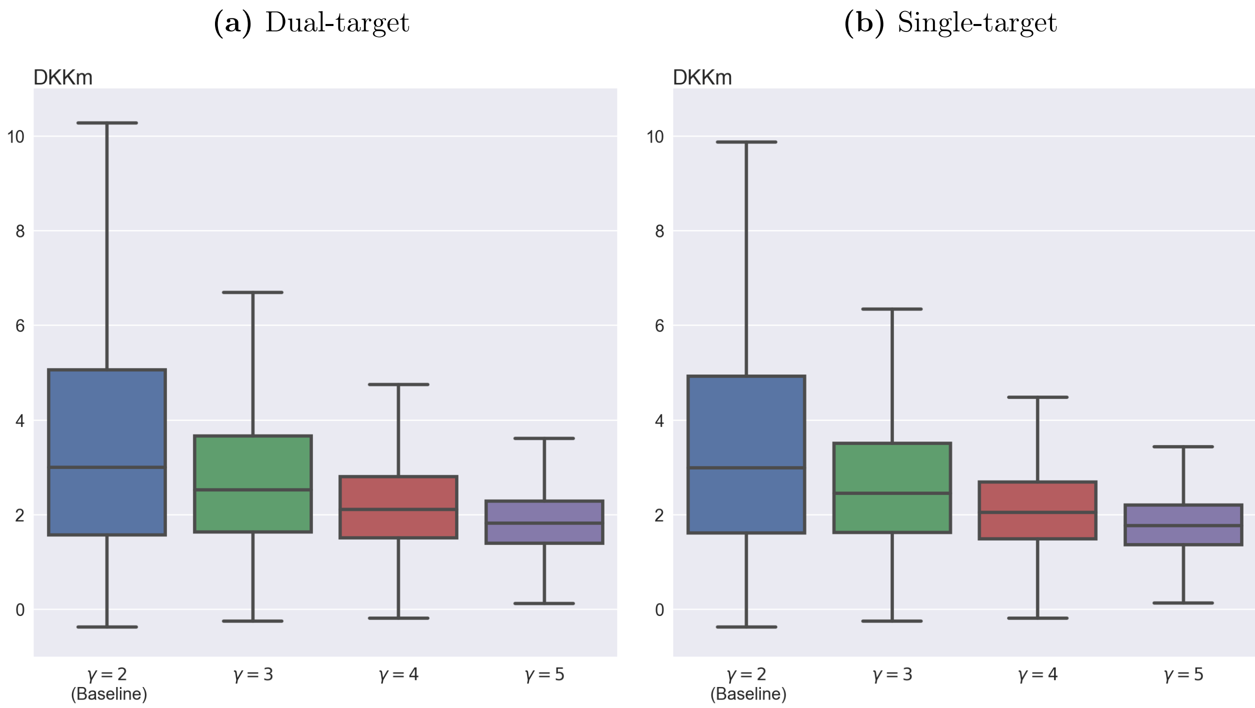


Figure 10: Terminal wealth when varying risk aversion γ



The lower risk can seemingly not outweigh the drop in returns. The relative risk aversion is not an especially useful measure when varying γ. Recall that we also used γ to calculate π, which determines the targeted stock market exposure. Changing γ would thus affect both stock market exposure and the measure of the certainty equivalent used for calculating the relative risk premium at the same time. This makes the measure unsuitable for comparisons across γ's.

To conclude, the terminal wealth increases as the level of risk aversion decreases by greater allocation in the risky asset. This likewise increases the default rate and the Sharpe ratio.

Table 5: Average and [median] Sharpe Ratios when varying γ

	Dual-target	Single-target	All-stock	Life-cycle
$\gamma = 2$	0.161[0.174] (0.090)	0.159[0.171] (0.090)	0.152[0.146] (0.088)	0.124[0.119] (0.092)
$\gamma = 3$	0.145[0.155] (0.077)	0.143[0.153] (0.077)	0.152[0.146] (0.088)	0.124[0.119] (0.092)
$\gamma = 4$	0.132[0.141] (0.070)	0.130[0.139] (0.070)	0.151[0.146] (0.088)	0.124[0.119] (0.092)
$\gamma = 5$	0.123[0.131] (0.064)	0.121[0.130] (0.063)	0.151[0.146] (0.088)	0.124[0.119] (0.092)

Note: Standard errors of the Sharpe Ratios in $(\cdot)$

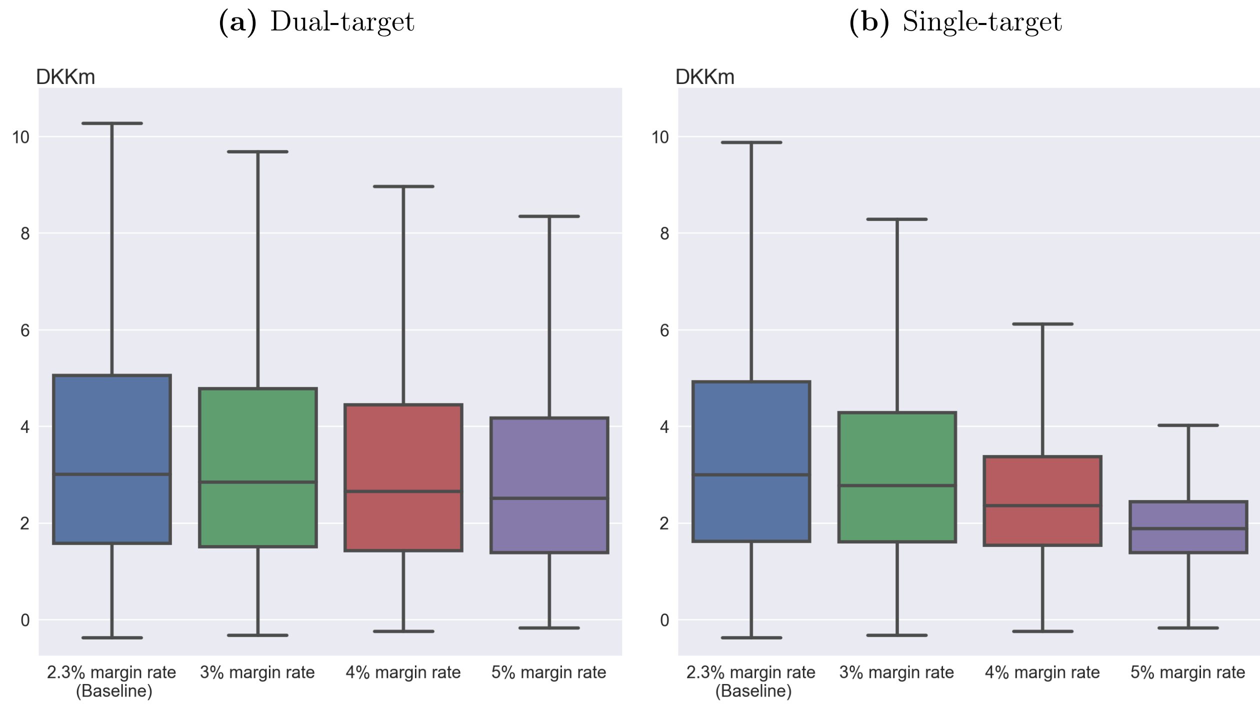
6.2.2 Steeper yield curve

The margin rate, r_m, is important for the leveraged strategies, both for determining interest costs and for setting the optimal allocation into the risky asset $\pi(r_m)$. Both channels have a major impact on the strategies' performance. One might expect the performance of the leveraged strategies to decrease drastically for a higher margin rate as borrowing money is more expensive. However, this is not the case, at least not for the dual-target strategy.

Figure 11: How does π change with increasing margin rates

In figure 11, we see the allocation into the risky asset falls as the margin rate increases. Intuitively, paying off the investor's debt is a better alternative to the risky asset when margin rates increase. We see that higher margin rates definitely harm both strategies, but more so for the single-target. This is because the dual-target strategy changes its allocation to depend on r_f in phase 3, which the single-target strategy does not. This has a dramatic effect on the single-target strategy. All the quantiles of terminal wealth in table 12 decrease rapidly as the margin rate increases, as does the Sharpe ratio, as seen in table 6. A similar trend

Figure 12: Terminal wealth when varying the margin rate



holds for the dual-target strategy, although much less profound. The Sharpe ratio barely falls, because when the investor has deleveraged, he increases his allocation to the risky asset to $\pi(r_f) = 98.7\%$. The dual-target strategy is, therefore, more robust against changes to the

Table 6: Average and [median] Sharpe Ratios and Relative Risk Premiums by varying margin rates

	Sharpe Ratio		Relative Risk Premiums	
	Dual-target	Single-target	Dual-target	Single-target
2.3% margin rate	0.161[0.174] (0.090)	0.159[0.171] (0.090)	102.26%	87.76%
3% margin rate	0.161[0.172] (0.088)	0.151[0.164] (0.086)	108.703%	61.09%
4% margin rate	0.160[0.170] (0.085)	0.140[0.152] (0.079)	102.313%	19.85%
5% margin rate	0.159[0.167] (0.081)	0.126[0.136] (0.071)	102.313%	-6.345%
	Sharpe Ratio		**Relative Risk Premiums**	
	All-stock	Life-cycle	All-stock	Life-cycle
All margin rates	0.152[0.146] (0.088)	0.124[0.119] (0.092)	132.68 %	50.13 %

Note: Standard errors of the Sharpe Ratios in $(\cdot)$

cost of borrowing money. Additionally, increases in the margin rate have little impact on the default risk, with 1.15%, 1.20%, 1.20%, and 1.15% investors defaulting with margin rates of 2.3%, 3%, 4%, and 5%.

Why do we not observe an increase in default rates? Increasing the margin rate will, ceteris paribus, cause *more* investors to default, but investors will also allocate less wealth into risky assets because of the Merton fraction, causing *fewer* investors to default. The two effects seem to cancel each other out roughly.

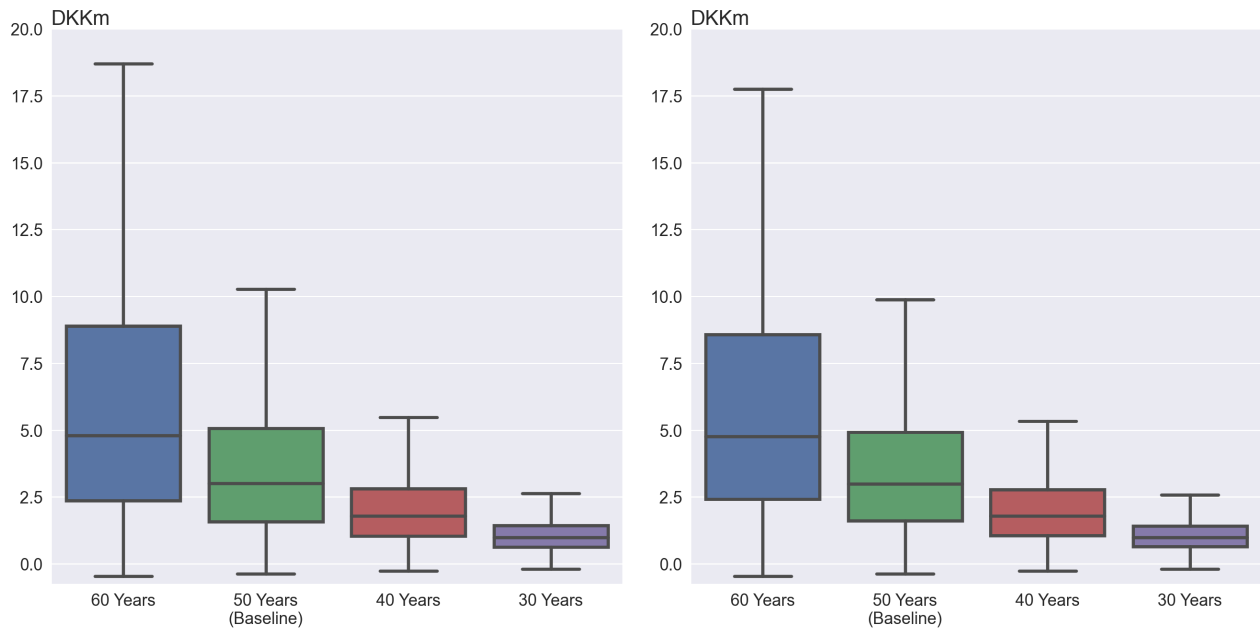
Finally, consider the relative risk premiums in table 6, where the meager allocation to stocks in the single-target strategy at high margin rates is evident. Here, the relative risk premium for the single-target strategy is even lower than for the life-cycle strategy at a margin rate of 4%.

6.2.3 Varying investment horizon

Our default investment horizon for the investor is 50 years, which is a very long horizon. Thus, an interesting question is whether the leveraged strategies are still viable for shorter horizons. When the horizon is shorter, the investor might not reach the last or even second-last stage of either strategy, which could be detrimental to the average performance. In figure 13, we

Figure 13: Terminal wealth of leveraged strategies by investment horizon

(a) Dual-target **(b)** Single-target



see that the volatility in terminal wealth increases greatly as the investment horizon increases. This is hardly surprising as the longer the investment horizon, the greater the number of feasible outcomes.

The risk of default increases linearly with the investment horizon, with 73, 91, 112, and 137 investors defaulting for 30, 40, 50, and 60 years respectively. This is likely due to investors who have been hit by significant negative returns when fully geared and are afterward struggling to pay off their debt. As time goes on, they will eventually default, which explains the increased number of defaults for longer time horizons.

Lastly, the dual- and single-target strategy barely differ for horizons of 30 and 40 years. Recall, that phase 3 of the dual-target strategy is to increase the allocation to the risky asset from $\pi(r_m)$ to $\pi(r_f)$ where the single-target sticks to $\pi(r_m)$. Thus, the dual-target strategy doesn't have enough time to gain a sizable advantage since the difference between $\pi(r_f)$ and $\pi(r_m)$ is relatively short in our baseline model. However, for horizons of 50 and 60 years, the dual-target strategy slightly outperforms the single-target strategy.

Consider now the performance measures in table 7. The relative risk premium favors the life-cycle strategy for all horizons tested. The all-stock strategy favors short horizons. The dual- and single-target outperform the all-stock in all but the 30-year horizon, with the dual-target relative risk premium increasing the most. In terms of Sharpe ratios, the dual- and single-target strategy outperforms the all-stock and life-cycle strategies for all horizons. This gap increases the shorter the investment horizon.

Table 7: Average and [median] Sharpe Ratios and Relative Risk Premiums by investment horizon

	Sharpe Ratio			
	Dual-target	Single-target	All-stock	Life-cycle
60 years	0.160[0.173] (0.087)	0.157[0.170] (0.086)	0.159[0.155] (0.079)	0.131[0.126] (0.085)
50 years	0.161[0.174] (0.090)	0.159[0.171] (0.090)	0.152[0.146] (0.088)	0.124[0.119] (0.092)
40 years	0.165[0.178] (0.095)	0.162[0.176] (0.094)	0.143[0.139] (0.095)	0.117[0.113] (0.099)
30 years	0.170[0.184] (0.102)	0.169[0.182] (0.101)	0.134[0.129] (0.107)	0.110[0.105] (0.110)
	Relative Risk Premiums			
	Dual-target	Single-target	All-stock	Life-cycle
60 years	121.51%	93.20%	176.31%	70.97%
50 years	102.26%	87.76%	132.68%	50.13%
40 years	70.25%	61.29%	78.58%	34.50%
30 years	55.69%	49.37%	48.77%	23.06%

Note: Standard errors of the Sharpe Ratios in $(\cdot)$

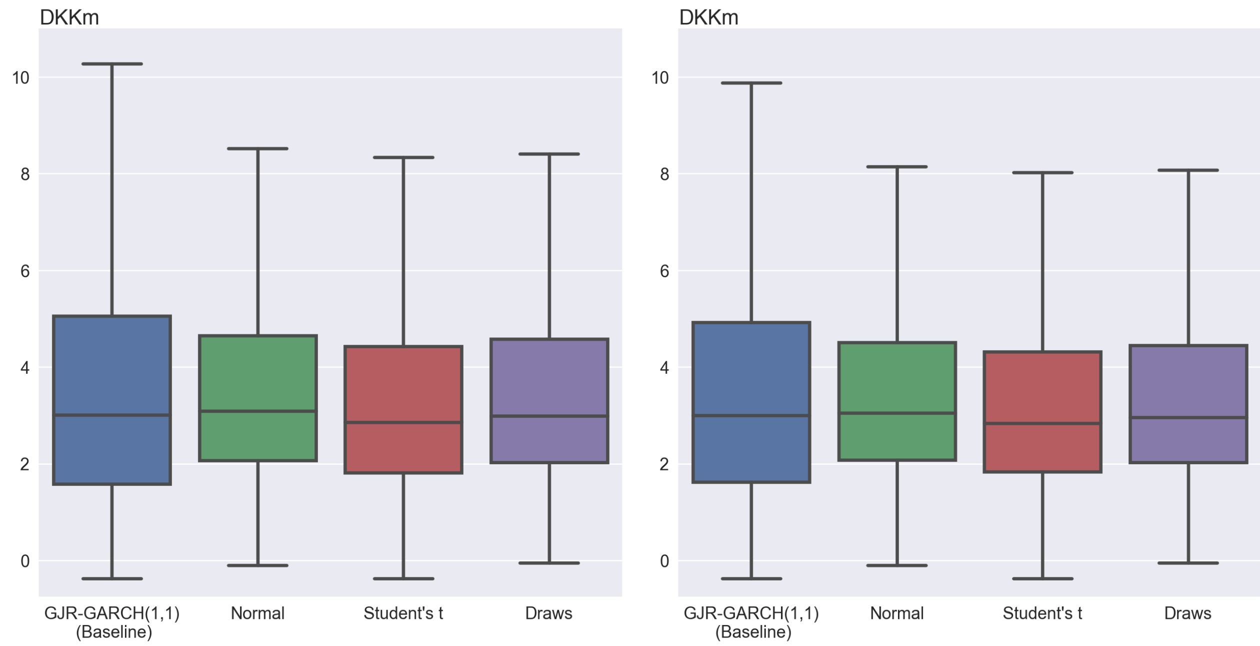
To summarize, the dual- and single-target are viable strategies for both short and longer horizons.

6.2.4 Simulation Method

A departure in this paper from similar literature is to use a GJR-GARCH(1,1) process to model the volatility of returns. Some previous literature assumes that market returns are IID, which is not empirically correct. As we argued in section 3, an empirical setup with skewed $- t_{\eta,\lambda}$ GJR-GARCH(1,1) is a more realistic data generating process (DGP) out of the ones presented. To test the significance of changing the data generating process, we compare the strategies simulated from a skewed $- t_{\eta,\lambda}$ GJR-GARCH(1,1) process to simpler and more naive processes like drawing returns from a fitted normal- or Student's t-distribution.

Figure 14: Terminal wealth of strategies by data generating process

(a) Dual-target **(b)** Single-target



From figure 14, it is clear that the simulation method has a major impact on the tails of the distribution of terminal wealth. When we draw returns from an empirical sample with replacement, or a normal distribution, no investors following the dual- and single-target strategies have terminal wealth below 0, which also means zero investors default in both cases. Our finding of no defaults aligns with the results from [Ayres and Nalebuff, 2008] where zero investors defaulted in their simulations with 'draws with replacement.' The reason for no defaults is the non-fat tails of the normal distribution and the IID returns of draws from replacement. These distributions cause fewer extreme events to happen (See figure 4), and investors are thus less likely to default due to unpayable debt.

Moving on to the case where the data generating process is an IID student's t distribution, several investors now have a negative terminal wealth and 78 investor default when using this data generating process. The IID student's t-distribution causes these defaults because of its fatter tails compared to the normal distribution or 'draws from replacement.' Comparing the IID student's t to the GJR-GARCH(1,1) process, we see that 112 investors default. Note

Table 8: Average and [median] Sharpe ratios and Relative risk premiums by data generating process

	Sharpe Ratio			
	Dual-target	Single-target	All-stock	Life-cycle
GJR-GARCH	0.161[0.174] (0.090)	0.159[0.171] (0.090)	0.152[0.146] (0.088)	0.124[0.119] (0.092)
Student's t	0.196[0.199] (0.064)	0.194[0.197] (0.064)	0.134[0.129] (0.076)	0.115[0.108] (0.078)
Normal	0.186[0.187] (0.049)	0.184[0.185] (0.048)	0.120[0.117] (0.068)	0.104[0.100] (0.070)
With replacement	0.194[0.196] (0.051)	0.192[0.195] (0.050)	0.123[0.119] (0.069)	0.106[0.102] (0.070)
	Relative Risk Premiums			
	Dual-target	Single-target	All-stock	Life-cycle
GJR-GARCH	102.26%	87.76%	132.68%	50.13%
Student's t	49.48%	44.68%	45.53%	30.92%
Normal	46.35%	42.62%	35.58%	24.65%
With replacement	47.68%	43.69%	36.71%	24.94%

Note: Standard errors of the Sharpe Ratios in (·)

that these two distributions are almost identical but have very different correlation structures. This difference means an additional 34 investors default by adding a more realistic correlation structure between returns. Furthermore, the 25^{th} quantile is noticeably lower, and the 75^{th} quantile is likewise higher for the GJR-GARCH(1,1) than for the student's t-distribution.

Lastly, consider the performance measurements in table 8. We see that adding the GJR-GARCH(1, 1) model in the data generating process significantly affects the relative risk premiums. The risk premium increases greatly for all but the life-cycle strategy. For all the non GARCH data generating processes, the dual- and single-target strategies have the highest risk premiums, possibly due to higher default rates and larger variations in terminal wealth. The dual- and single-target strategies yield a higher Sharpe ratio across all four data-generating processes tested, indicating that the leveraged strategies outperform on a risk-adjusted basis regardless of the data generating process. The GJR-GARCH process yields the lowest average Sharpe ratio of all the different processes, likely due to its higher default rate.

To conclude, the choice of data generating process for the returns has a significant impact on the default rate, which the addition of the GJR-GARCH(1,1) model increased to 1.12% from 0%. Thus, default risk is a bigger problem than [Ayres and Nalebuff, 2008] argued. Furthermore, the choice of data generating process has a big impact on performance measures, with the Sharpe ratio falling quite drastically if returns are not IID.

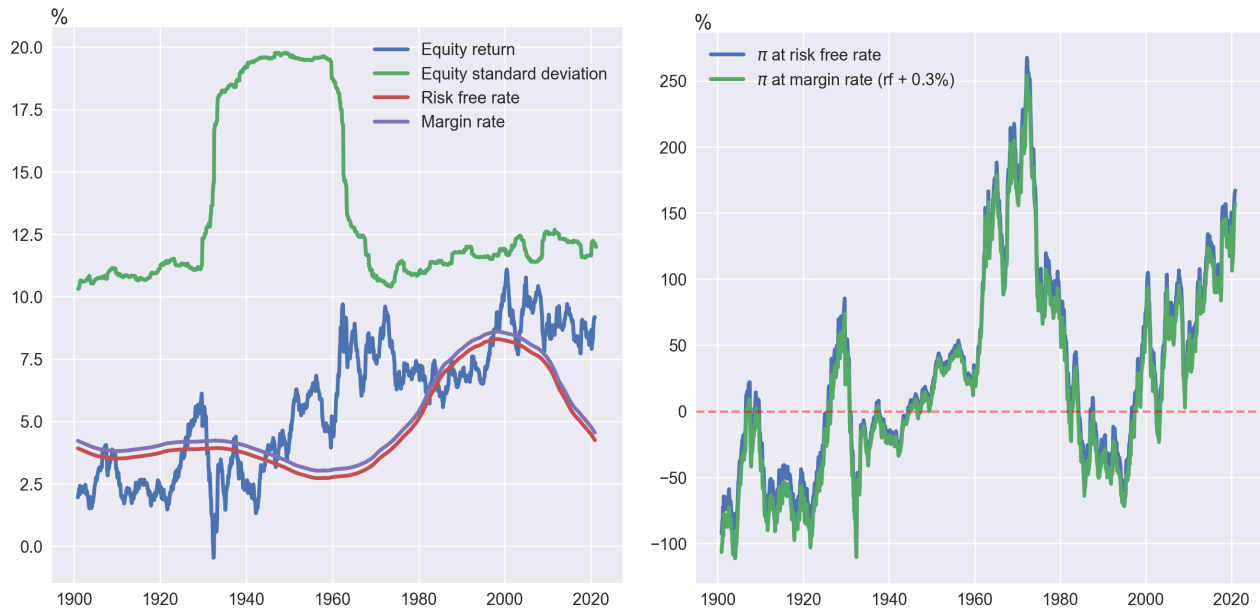
Additionally, for the GARCH and student's t process, the distribution of terminal wealth is shifted slightly downwards compared to the normal distribution or draws with replacement despite having similar mean returns with lower 25$^{\text{th}}$ quantile. This is yet again a result of the fat tails of the student's t-distribution.

6.3 Backtesting using historical data

A natural question arises when evaluating performance. How would the model have performed in the past, given the realized markets of the past century? To test this, we plug in data from Robert Shiller's website [Shiller, 2021]. Similar to the method we used in simulating future returns, we also calculate a 30-year rolling average of market returns, market volatility, and the risk-free rate to calculate historical $\pi(r_f)$ and $\pi(r_m)$.

Figure 15: Using Robert Shillers data

(a) Rolling 30 year averages of macro variables **(b)** Resulting $\pi(r_f)$ and $\pi(r_m)$

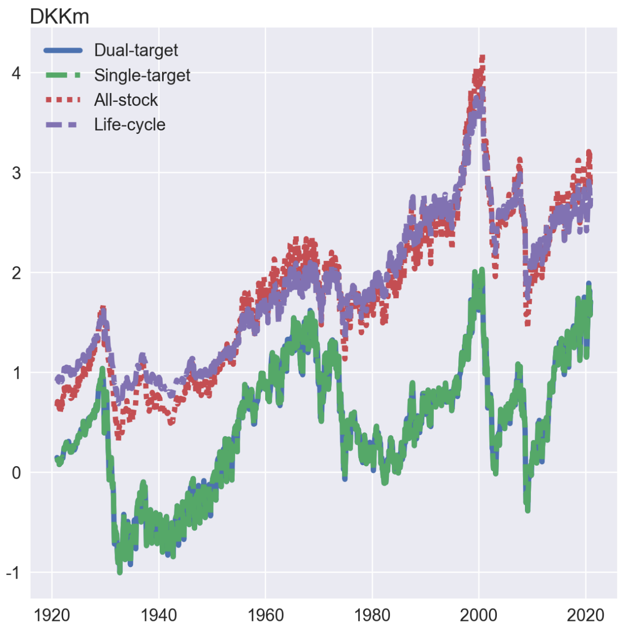


As is evident from figure 15, the strategy breaks down had it been calibrated using a rolling window of 30 years during many parts of the previous century. Particularly problematic are the many periods with negative values of π. A negative π implies building a short position in equities - a questionable strategy for a leveraged long-term investor. One possible remedy for this could be to run the strategy with the average values of the entire sample period from 1871 to 2020. Doing this, we get $\pi(r_f) = 29.71\%$ and $\pi(r_m) = 20.09\%$. Another way around this could be to enforce our baseline values of $\pi(r_f) = 98.70\%$ and $\pi(r_m) = 93.41\%$ irrespective of the actual market conditions at the time. To test the strategy, we simulate a 50-year strategy from January 1871 to January 1921 and roll this window forward until the present day. By doing this, we obtain 1200 simulations. We plot the terminal wealth for every simulation of both approaches below:

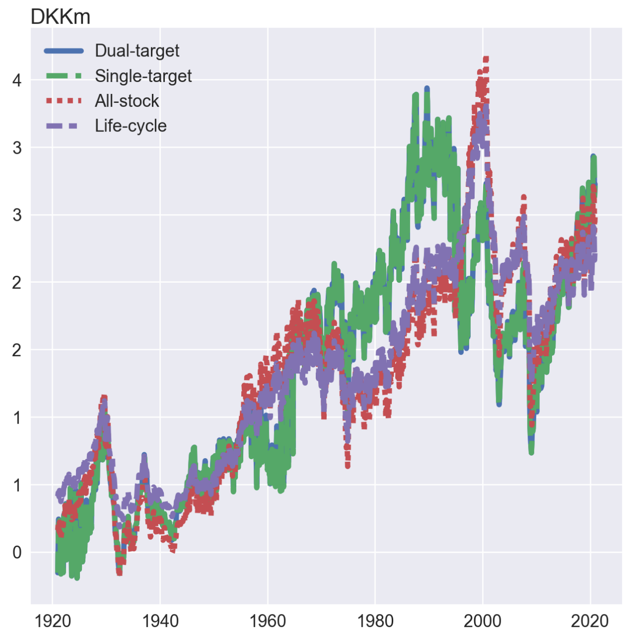
On the left-hand chart, we note that at no point do the leveraged strategies outperform the unleveraged strategies. This is probably due to the very conservative stock exposure of around

Figure 16: Terminal wealth of strategies by ending year with historical backtesting

(a) $\pi(r_f) = 29.71\%$ and $\pi(r_m) = 20.09\%$ (Historial averages)



(b) $\pi(r_f) = 98.70\%$ and $\pi(r_m) = 93.41\%$ (Baseline)



25%. On the right-hand side, we see some periods, especially from 1965 until 1995, with the leveraged strategies outperforming the unleveraged strategies. However, there are also periods with performance on par or below the traditional strategies. The average outperformance of the levered strategies is around DKK 50.000, so not a large average gain by any means. Interestingly, we find that no investors would have defaulted during this period, regardless of the two sets of π's we use. This is identical to the findings of [Ayres and Nalebuff, 2008] who also report zero defaults in their cohort simulations. However, contrary to [Ayres and Nalebuff, 2008], We don't find that the levered strategies outperform the unleveraged strategies in all years. This is likely due to a different margin rate.

Table 9: Average and [median] Sharpe Ratios and Relative Risk Premiums of historical backtest using baseline $\pi(r_f)$ and $\pi(r_m)$

	Dual-target	Single-target	All-stock	Life-cycle
Sharpe Ratio	0.131[0.152] (0.054)	0.130[0.151] (0.054)	0.096[0.097] (0.030)	0.082[0.087] (0.029)
Relative Risk Premium	36.54%	36.73%	31.08%	18.69%

Note: Standard errors of the Sharpe Ratios in $(\cdot)$

The leveraged strategies perform well regarding risk-adjusted returns when looking at the Sharpe ratios, similar to our findings in the baseline model simulations. We find Sharpe ratios around 0.130 for the leveraged strategies - slightly higher than the Sharpe ratio for the all-stock strategy of 0.096 and 0.082 for the life-cycle strategy. In terms of relative risk premium, the leveraged strategies are only deemed slightly more risky by the risk-averse investors than the all-stock strategy.

7 Discussion

The curious reader might be slightly puzzled as to why our risky asset is built on S&P 500 returns instead of a Danish index like the OMXC25 index when investors worldwide exhibit strong home bias, i.e., they favor local stocks.

The reasons for choosing the S&P 500 are twofold. Firstly, to more closely compare our results to [Ayres and Nalebuff, 2008] we use the same market index as them. Secondly, the S&P 500 is a broad index with 500 stocks and is thus more diversified than OMXC25, which only contains 25 stocks concentrated around biotech/health and industry. If we used the OMXC25, we could risk that the strategy's returns stem from the development in these industries rather than the market as a whole. Following this logic, why not use an even more diversified global index, say, the MSCI world index, as a reference? We have a few reasons. First, the MSCI world index and other global indices mainly consist of US large-cap stocks, so there is actually a significant overlap with the S&P 500. Second, the S&P 500 has a far longer history allowing for more reliable estimates of the statistical properties. Third, the S&P 500 is broadly accepted as a close proxy for the market portfolio in the literature.

Another critique of our results might be that we have assumed zero trading costs. However, we think it is a reasonable assumption. First, we recommend investments in passive liquid ETF's tracking the S&P 500 or a similar broad index. This means spread costs are virtually eliminated. Second, commission-free brokers are becoming more and more widespread, with several available in the US and EU. In other words, it is so cheap to be a passive index investor today, so even if we included a small fee, we would not expect it to significantly alter our results, even in the strategies with monthly rebalancing.

The combination of leverage and risk-averse investors initially sounds counter-intuitive. Why would a rational investor, who dislikes risk, borrow money to invest in stocks, effectively shorting the risk-free asset to long stocks? It could be a sensible idea for a few reasons. Firstly, although a 1:1 gearing ratio of current savings might seem high to a young investor, the amount of money borrowed is insignificant compared to his cumulative life savings. Secondly, gearing is not permanent and is used to even out exposure to the stock market over time. In this way, gearing might decrease risk, which is in line with being a risk-averse investor.

Continuing with the theme of a risk-averse investor. We only consider cases for the constant relative level of risk aversion, $\gamma \in \{2, 3, 4, 5\}$, such that the investor is moderate to highly risk-averse. One might reasonably argue that we should consider cases where γ was lower, like $\gamma = 1$ resulting in investors having log utility or even $\gamma = 0$ where investors are risk-neutral. The reasons for not considering these cases are twofold. First, investors are empirically risk-averse, so modeling risk-neutral investors would not be as relevant as modeling risk-averse investors.

Secondly, for lower values of γ like $\gamma = 1.5$, we have that $\pi(r_i) > 1$ such that investors would never pay off all their debt while investing. While there is not anything inherently wrong with this, the focus of this paper is whether an investor can use leverage to achieve higher risk-adjusted returns. Thus, cases where the investor is leveraged throughout his life-cycle are not within this paper's scope, so we do not consider lower values of γ.

Compared to [Ayres and Nalebuff, 2008] we have left out discounting in our performance measurement. One could argue that discounting the strategies would make comparing the choices today easier. However, since we are primarily interested in the relative performance of the strategies, discounting didn't offer any benefits, as discounting, at any rate, would never qualitatively change our results. Due to this, we decided not to prioritize space on discounting.

One might argue that our results for risk-averse investors rest upon CRRA utility. But we primarily use the CRRA utility function to derive the optimal allocation π^*. CRRA preferences are just one way to model the risk-aversion of investors. Another way is a more simple mean-variance utility function. Using this utility function to get the optimal allocation π^*, the π^* might change as compared to the CRRA approach. However, it would not change the market dynamics and diversification principles. Thus the main result of temporal diversification and its benefits to young investors would remain largely unchanged.

Lastly, we use a risk-free rate, r_f of 2%, roughly the historical average yield of a 10-year Danish government bond. Given the current rate of a Danish government bond, one might argue that our risk-free rate r_f should be closer to 0%. We argue that the risk-free rate being 0% for most of the next 40-60 years is unlikely. Therefore, we choose the average yield of government bonds, which is also standard in the literature. In any case, what constitutes a natural or equilibrium level for the risk-free rate is a topic of continuous research, and no definitive right answer exists. Thus, forecasting the risk-free rate decades from now is nearly impossible.

In the parameterization of our baseline model, we use the 15-year historical average to determine the risk-free rate r_f but use current rates to determine the margin rate r_m. This seems inconsistent. However, we argue that the two rates are used in two different time horizons. r_f affects the strategy in all periods, while r_m is only relevant for the first 15-20 years in the majority of our simulations. We, therefore, chose to bias the estimate of r_f toward historical averages and r_m toward present rates. Even if our estimate of r_m is too low, we also cover the case of a steeper yield curve in the analysis and find that the leveraged strategies still outperform the rest. If our estimate of r_f is too high, the leveraged strategies actually perform better, as the lower risk-free rate is detrimental to the life-cycle strategy. At the same time, it increases stock exposure in the leveraged strategies. Simulations also confirm this, but we were limited by the space available to show it.

8 Conclusion

In this paper, we model leveraged investment strategies for young Danish students in a market where a constant mean GJR-GARCH(1,1) with skewed $t_{\eta,\lambda}$-distributed error terms generates simulated returns. We test leveraged dual- and single-target strategies and find that they yield higher terminal wealth than the unleveraged all-stock and life-cycle strategies in 95% of simulated markets. The performance somewhat persists in the risk-adjusted measure, where the mean Sharpe ratio is 0.161 and 0.159 for the dual- and single-target strategies compared to the 0.152 and 0.123 for the all-stock and life cycle strategies. The relative risk premiums favor the life-cycle strategy and deem the all-stock single the riskiest, with investors demanding the highest risk premium of 132.68% for the all-stock strategy. The dual- and single-target strategies lie in the middle with 102.26% and 87,76%, respectively. These results are in slight contrast to the results of [Ayres and Nalebuff, 2008], who find that the dual- and single-target strategies stochastically dominate the more traditional all-stock and life-cycle strategies. This is likely due to our usage of a more realistic data-generating process with fatter tails.

There are several leverage options available to young investors, some more attractive than others. Students loan (SU loans) combined with portfolio loans seem the most attractive and available option with a low effective interest rate of around 2.3%. As such, a leveraged strategy is entirely feasible for young students in Denmark.

The dual- and single-target strategies spread out exposure to stocks as the leverage taken allows for more even exposure to the risky asset. The risk-reducing effect from temporal diversification outweighs the additional risk from leveraging and the possibility of default, where 1.12% of investors default in the baseline setting. This is evident from the higher Sharpe ratios of the leveraged strategies.

From 50-year rolling window backtests of the strategies, we conclude that the leveraged strategies would historically have faired well on a risk-adjusted basis, judging by the higher Sharpe ratio compared to unleveraged strategies. When comparing the strategies on average terminal wealth, the leveraged strategies only come out slightly on top. Importantly, zero investors would have defaulted during our sample period from 1871 to 2021.

Investors, who default, are usually hit by highly adverse market returns around their period of maximum indebtedness such that the servicing of their debt causes them to default. The default rate is very dependent on the data generating process as when returns are 'drawn with replacement' or a fitted normal distribution, no investor defaults.

Would we recommend the dual- and single-target strategies to young, risk-averse, danish students? Based on the results from our simulations and the above conclusion, yes. However, the dual- and single-target strategies do have some additional drawbacks. They are more complex with their frequent rebalancing compared to the all-stock and partially the life-cycle strategies. A young, risk-averse investor needs to be willing to spend time managing the strategy. We recommend a more simple single-target strategy for an investor unwilling to spend the time required to manage the dual-target strategy. Even though we have shown that the strategies are robust against a rise in margin rates and differing data generating processes, this recommendation is still conditional on our expectation of the risk-free rate, the margin rate, and the development of the market, which we cannot know with certainty.

If the leveraged strategies seem too complicated or risky, it is, in any case, a good idea to start investing early, if just in a normal all-stock strategy. As shown in our theoretical section, the benefits of time diversification are present regardless of using our leveraged strategy or not - the leveraged strategies merely intensify them.

The model driving simulated market returns is a GJR-GARCH model with skewed $t_{\eta,\lambda}$-distributed error terms. This allows for one risky asset. However, the market consists of multiple different stocks and other asset classes. A natural avenue for future research is to allow for other assets and use a Multivariate GARCH model (MGARCH) to model their returns.

Another idea for future research is to leave the Merton fraction and use a more complex allocation method by making $\pi(\cdot)$ dynamic over time. This will allow for greater flexibility of the investment strategies, which could offer new potential benefits.

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A Appendix

A.1 Skewed Student's t-distribution

The following density characterizes the Skewed Student's t-distribution

$$F(z|\eta,\lambda) = \begin{cases} bc\left(1+\frac{1}{\eta-2}\left(\frac{bz+a}{1-\lambda}\right)^2\right)^{-\frac{\eta+1}{2}} & z < -a/b \\ bc\left(1+\frac{1}{\eta-2}\left(\frac{bz+a}{1+\lambda}\right)^2\right)^{-\frac{\eta+1}{2}} & z \geq -a/b \end{cases}, \tag{X}$$

and a, b and c are given by

$$a = 4\lambda c\left(\frac{\eta-2}{\eta-1}\right), \qquad b = \sqrt{1+3\lambda^2-a^2}, \qquad c = \frac{\Gamma\left(\frac{\eta+1}{2}\right)}{\sqrt{\pi(\eta-2)}\Gamma\left(\frac{\eta}{2}\right)}. \tag{X}$$

λ dictates the level of skewness and for $\lambda > 0$ the distribution is skewed to the right.η determines the fatness of the tails and the number of moment which are finite as such $\eta > 2$ for the variance to be finite and $\Gamma(\cdot)$ is the Gamma function. Note that for $\eta \to \infty$ the student t distribution converges to a normal distribution.[4]

A.2 Python Code

All the code used to generate the simulations and plots is freely available at
`https://github.com/neriksen/applied_finance`

A.3 Parameters for polynomial fit of disposable income

$$c = 9000, slope = 0.014885, convexity = -0.0000373649, jerk = 0.000000025$$

$$p(x) = c + x \cdot slope \cdot c + x \cdot convexity \cdot c + x \cdot jerk \cdot c \tag{X}$$

$$p(x)_{infl} = c + x \cdot slope \cdot c \cdot (0.03/12) + x \cdot convexity \cdot c + x \cdot jerk \cdot c \tag{X}$$

Where c is the constant, interpreted as the monthly disposable income as a 20 year old and x is the month starting age 20 being equal to $x = 0$

[4][Hansen, 1994], *Autoregressive conditional density estimation*, p.708-709